Mixed Hodge Structures on Cohomology Jump Ideals

Periods, Shafarevich Maps and Applications

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Introduction

- X : smooth algebraic variety $/\mathbb{C}$ ($X \hookrightarrow \overline{X}$ proper)
  or : $X \hookrightarrow \overline{X}$ compact Kähler (X : *quasi-Kähler*)
  with complement a normal crossing divisor
- $x \in X$: base point
- $\pi_1(X, x)$: fundamental group, finitely presentable
- $G \subset GL(N)$: linear algebraic group $/\mathbb{C}$
- $\mathfrak{g} := \mathrm{Lie}(G) \subset \mathrm{End}(\mathbb{C}^N)$

General Topic

Study the topology of X *via* linear representations of $\pi_1(X, x)$ into G.

Moduli space of representations

$$\mathrm{Hom}(\pi_1(X,x),G) := \underset{\textbf{Group}}{\mathrm{Hom}}(\pi_1(X,x),G(\mathbb{C})) \tag{1}$$

$+$ structure of affine scheme of finite type $/\mathbb{C}$
$=$ moduli space of representations ρ of $\pi_1(X,x)$ into G
$=$ moduli space of G-local systems V_ρ over X with a framing $V_{\rho,x} \simeq \mathbb{C}^N$ at x.

Cohomology jump loci

$k, i \in \mathbb{Z}_{\geq 0}$

$$\Sigma^i_k = \left\{\rho : \pi_1(X,x) \longrightarrow G(\mathbb{C}) \mid \dim(H^i(X,V_\rho)) \geq k\right\} \subset \mathrm{Hom}(\pi_1(X,x),G) \tag{2}$$

closed subschemes.

Question : local and global structure of $\mathrm{Hom}(\pi_1(X,x),G)$ and the Σ^i_k
$\rightsquigarrow$ restrictions on the topology of X.

Examples of known results

$G = \mathbb{C}^*$:

$$\mathrm{Hom}(\pi_1(X,x), \mathbb{C}^*) \simeq (\mathbb{C}^*)^{b_1(X)} \times (\text{finite}) \tag{3}$$

- Local structure: Esnault-Schechtman-Viehweg: X complement of an arrangement of hyperplanes, $G = \mathbb{C}^*$, at $\rho = 1$,
  germ at ρ of $\Sigma^i_k \simeq$ germ at 0 of

$$\left\{\omega \in H^1(X,\mathbb{C}) \mid \dim H^i\Big(H(X,\mathbb{C}), \omega \wedge -\Big) \geq k\right\} \tag{4}$$

  Generalizations of Dimca-Papadima-Suciu in rank N
- Irreducible components of Σ^i_k passing through ρ are sub-tori translated by torsion points (successive generalizations : Beauville, Green-Lazarsfeld, Arapura, Simpson, Budur-Wang, Budur-Rubió. . .)
- Esnault-Kerz (arithmetic methods), Budur-Wang, Lerer: X algebraic $/K \subset \mathbb{C}$, Σ^i_k is *motivic*: $T_\rho \mathrm{Hom}(\pi_1(X,x), \mathbb{C}^*) \subset H^1(X,\mathbb{C})$ sub-MHS and *absolutely constructible*
  $\rightsquigarrow$ arithmetic properties, geometric origins.

Main object of study

$\rho : \pi_1(X,x) \longrightarrow G(\mathbb{C}) \longleftrightarrow$ point $\rho \in \mathrm{Hom}(\pi_1(X,x), G)$

Definition (Cohomology jump ideals)

- $\widehat{\mathcal{O}}_\rho =$ completed local ring of $\mathrm{Hom}(\pi_1(X,x), G)$ at ρ
- $J^i_k \subset \widehat{\mathcal{O}}_\rho =$ ideal defining Σ^i_k at ρ

Interest: for a local Artin algebra A $/\mathbb{C}$ (ex: $A = \mathbb{C}[t]/t^n$)

$$\mathrm{Def}(\rho)(A) := \mathrm{Hom}(\widehat{\mathcal{O}}_\rho, A) = \left\{ \tilde{\rho} : \pi_1(X,x) \longrightarrow G(A) \mid \tilde{\rho} = \rho \text{ over } G(\mathbb{C}) \right\} \quad (5)$$

$\rightsquigarrow$ deformation function $\mathrm{Def}(\rho) : \mathbf{Art} \longrightarrow \mathbf{Set}$.

Similarly : $\mathrm{Def}^i_k(\rho) : A \mapsto \mathrm{Hom}(\widehat{\mathcal{O}}_\rho / J^i_k, A)$

Remark: everything is defined over $\mathbf{k} \subset \mathbb{C}$ if G, ρ are defined over $\mathbf{k}$.

Main result

Theorem (Eyssidieux-Simpson, 2008)

X compact, V_ρ underlying a VHS: construction of a MHS on $\widehat{\mathcal{O}}_\rho$.

Theorem (L., 2019, 2020)

Extension to X non-compact, V_ρ underlying an admissible VMHS with unipotent monodromy at infinity.

Theorem (L., 2021)

The $J^i_k \subset \widehat{\mathcal{O}}_\rho$ are sub-MHS.

Tool: differential graded Lie algebras

X smooth, ρ fixed, over $\mathbb{C}$.
$\rightsquigarrow$ flat principal G-bundle P_ρ, adjoint bundle $\mathrm{ad}_\rho = P_\rho \times_G \mathfrak{g} \subset \mathcal{E}nd(V_\rho)$.
$\rightsquigarrow$ $L := \mathscr{E}^\bullet_{\mathcal{C}^\infty}(X, \mathrm{ad}_\rho)$ equipped with $d : L^i \to L^{i+1}$ and $[-,-] : L^i \times L^j \to L^{i+j}$: differential graded Lie algebra.
$\rightsquigarrow$ $M := \mathscr{E}^\bullet_{\mathcal{C}^\infty}(X, V_\rho)$ equipped with $d : M^i \to M^{i+1}$ and $L^i \times M^j \to M^{i+j}$: module over L.
$+\ \varepsilon_x : L \to \mathrm{ad}_{\rho,x} \simeq \mathfrak{g}$ augmentation at x

Deformation functors

L: any DG Lie algebra over $\mathbf{k}$ $\rightsquigarrow$ deformation functor $\mathrm{Def}(L)$. For $(A, \mathfrak{m}) \in \mathbf{Art}$:

$$\mathrm{Def}(L)(A) := \left\{ \omega \in L^1 \otimes \mathfrak{m} \;\middle|\; d(\omega) + \frac{1}{2}[\omega, \omega] = 0 \right\} / \exp(L^0 \otimes \mathfrak{m}) \qquad (6)$$

For (L, M) any DG Lie pair $\rightsquigarrow$ sub-functors $\mathrm{Def}^i_k(L, M) \subset \mathrm{Def}(L)$:
for $\omega \in \mathrm{Def}(L)(A)$ $\rightsquigarrow$ differential $d_\omega := d \otimes \mathrm{id}_A + \omega$ on $M \otimes A$
$\rightsquigarrow$ jump ideals $J^i_k(M \otimes A, d_\omega) \subset A$
$\rightsquigarrow$ $\mathrm{Def}^i_k(L, M)(A) := \left\{ \omega \in \mathrm{Def}(L)(A) \;\middle|\; J^i_k(M \otimes A, d_\omega) = 0 \right\}$.

Theorem

- *(Main principle)* $L \to L'$ *quasi-isomorphism of DG Lie algebras* $\Longrightarrow$ $\mathrm{Def}(L) \to \mathrm{Def}(L')$ *isomorphism of deformation functors.*
- *(Goldman-Millson) Let* $L_0 := \mathrm{Ker}(\varepsilon_x : L \to \mathfrak{g}) \Longrightarrow \mathrm{Def}(L_0) \simeq \mathrm{Def}(\rho)$.
  $\Longrightarrow \widehat{\mathcal{O}}_\rho$ *is determined by* (L, ε), *up to quasi-isomorphism of* L.
- *(Budur-Wang)* $(L, M) \to (L', M')$ *quasi-isomorphism of DG Lie pairs* $\Longrightarrow$ $\mathrm{Def}^i_k(L, M) \to \mathrm{Def}^i_k(L', M')$ *isomorphism;*
  and here $\mathrm{Def}^i_k(L_0, M) \simeq \mathrm{Def}^i_k(\rho)$.
  $\Longrightarrow J^i_k \subset \widehat{\mathcal{O}}_\rho$ *is determined by* (L, M, ε), *up to quasi-isomorphism of* (L, M).

Examples:

- X compact Kähler manifold, $G = GL(N)$, V_ρ polarized VHS (and $\mathrm{ad}_\rho \subset \mathcal{E}nd(V_\rho)$ also VHS)
  $\Longrightarrow$ (analytic methods) (L, M) is *formal*, i.e. quasi-isomorphic to $(H(L), H(M))$.
  $\Longrightarrow \widehat{\mathcal{O}}_\rho \approx$ complete local ring at 0 to the quadratic cone $[\omega, \omega] = 0 \subset H^1(X, \mathrm{ad}_\rho)$.
  $\Longrightarrow J^i_k \subset \widehat{\mathcal{O}}_\rho =$ ideal defining the ω s.t. $\dim H^i\Big(H(X, V_\rho), \omega\Big) \geq k$.
- $G = \mathbb{C}^*$, $\rho = 1$: the $\omega \in H^1(X, \mathbb{C})$ s.t. $\dim H^i\Big(H(X, \mathbb{C}), \omega \wedge -\Big) \geq k$

L_∞ algebras

In our cases X: non-compact, there are MHS on cohomology with coefficients in a VMHS, also on $H(L)$ and $H(M)$, but...
L may not be formal

Theorem (Derived deformation theory)

- *$H(L)$ has a structure of L_∞ algebra, with maps $\ell_n : H(L)^{\otimes n} \to H(L)$ of degree $2-n$, for all $n \geq 1$ ($\ell_1 = d = 0$, $\ell_2 = [-,-]$, ℓ_3: Massey product),*
- *L becomes quasi-isomorphic to $H(L)$ as L_∞ algebra.*
- *L is formal $\iff \ell_n = 0 \; \forall n > 2$.*
- *Any L_∞ algebra has a deformation functor, invariant under quasi-isomorphisms: for $(A, \mathfrak{m}) \in$ **Art***

$$\mathrm{Def}(H(L))(A) := \left\{ \omega \in H^1(L) \otimes \mathfrak{m} \;\middle|\; \sum_{n=1}^{+\infty} \frac{\ell_n(\omega, \ldots, \omega)}{n!} = 0 \right\} / \mathit{homotopy} \tag{7}$$

And for the pair (L, M):

Theorem (Budur-Rubió)

$H(M)$ has as structure of L_∞ module over $H(L)$, with maps

$$\mu_n : H(L)^{\otimes(n-1)} \otimes H(M) \to H(M) \tag{8}$$

of degree $2-n$, $n \geq 1$ ($\mu_1 = d_{H(M)} = 0$, $\mu_2 =$ action of $H(L)$ on $H(M)$) and (L, M) becomes quasi-isomorphic as L_∞ pair to $(H(L), H(M))$.
Any such L_∞ pair has deformation functors, invariant under quasi-isomorphisms of the pair: $\omega \in \mathrm{Def}(H(L))(A)$ defines a differential

$$d_\omega := \sum_{n\geq 1} \frac{1}{n!}(\mu_n \otimes A)(\omega, \ldots, \omega, -) \tag{9}$$

on $H(M) \otimes A$ and jump ideals $J^i_k(H(M) \otimes A, d_\omega) \subset A$, and

$$\mathrm{Def}^i_k(H(L), H(M))(A) := \Big\{\omega \in \mathrm{Def}(H(L))(A) \mid J^i_k(H(M) \otimes A, d_\omega) = 0\Big\}. \tag{10}$$

Come back to our problem:

Goal

ρ: representation such that V_ρ underlies a VMHS ($\Rightarrow \mathrm{ad}_\rho \subset \mathcal{E}nd(V_\rho)$ also), admissible
$\implies H(X, \mathrm{ad}_\rho), H(X, V_\rho)$ have MHS on cohomology and with bracket, action, being morphisms of MHS.
Goal: $\widehat{\mathcal{O}}_\rho$ has a MHS and J^i_k is a sub-MHS.

Problems:

- MHS exist only *on the cohomology* of (L, M) but the deformation functors depend on the entire (L, M) up to quasi-isomorphisms.
- Similarly: the Lie bracket and action maps over $\mathbb{Q} \subset \mathbb{R}$ exist only *on the cohomology* of L, M; the classical theories produce *cochain complexes* over $\mathbb{Q}$ and $\mathbb{C}$, quasi-isomorphic to L, M, equipped with filtrations, forming the data of a *mixed Hodge complex*, inducing the MHS on cohomology.
$\implies$ but we want *algebras over* $\mathbb{Q}$ *commutative at the level of cochains* (problem of rational homotopy theory) combined with Hodge theory.
- Deal with the augmentation ε_x: L may not have $H^0(L) = 0$ ($\mathrm{Def}(L)$ not pro-representable), but $\mathrm{Ker}(\varepsilon_x)$ does.

How to get these

Recall the following from classical Hodge Theory:

Theorem

- *(Deligne) X complex algebraic (or: quasi-Kähler) $\Longrightarrow$ all groups $H^n(X,\mathbb{Q})$ have a MHS.*
- *(Saito) V: admissible VMHS over $X \subset \overline{X} \Longrightarrow$ each $H^n(X,V)$ has a MHS.*

In this last case: $j : X \hookrightarrow \overline{X}$, $D = \overline{X} \setminus X$,
the data $(Rj_*\mathbb{Q}_{\underline{X}}, \Omega_{\overline{X}}(\log D) \otimes V)$ forms a *mixed Hodge complex*
(extend first V as $\mathcal{C}^\infty$ vector bundle)

And from Rational Homotopy Theory:

Theorem

- *(Morgan, Hain) Construction of MHS on $\pi_n(X) \otimes \mathbb{Q}$.*

This last constructions uses *multiplicative cochain complexes over* $\mathbb{Q}$ that compute $H(X,\mathbb{Q})$, using algebras of rational polynomial forms on X.

Plan of proof

- Previous work $\Longrightarrow$ construct (L, M) which is at the same time a DG Lie pair computing $(H(X, \mathrm{ad}_\rho), H(X, V_\rho))$, and equipped with filtrations, forming a mixed Hodge complex, inducing the MHS on cohomology.
  Components over $\mathbb{C}$: $\Omega_{\overline{X}}(\log D) \otimes \mathrm{ad}_\rho$ and $\Omega_{\overline{X}}(\log D) \otimes V_\rho$
  Components over $\mathbb{Q}$: rational polynomial forms with twisted coefficients
  Easy with the *Thom-Whitney resolution functors*.
- Recent work of Cirici-Sopena $\Longrightarrow$ $(H(L), H(M))$ structure of L_∞ pair with all higher operations being morphisms of MHS.
- The problem becomes linear algebra in finite-dimensional vector spaces equipped with MHS!
  $\Longrightarrow$ $\widehat{\mathcal{O}}_\rho$ has a MHS (Maurer-Cartan equation with MHS)
- For A: MH Artin ring, $\omega \in \mathrm{Def}(H(L))(A)$: MH Maurer-Cartan element (s.t. multiplication by ω is a morphism of MHS)
  - d_ω is a differential on $H(M) \otimes A$ compatible with MHS,
  - $J^i_k(H(M) \otimes A, d_\omega) \subset A$ is a sub-MHS,

  $\Longrightarrow$ $J^i_k \subset \widehat{\mathcal{O}}_\rho$ is a sub-MHS.

Application to the global structure

- X: quasi-Kähler
- $G = \mathbb{C}^*$
- W: admissible VMHS over X with unipotent monodromy at infinity

Consequences of splitting the weight filtration over $\mathbb{C}$:

Theorem (L., 2021)

The irreducible components of the relative cohomology jump loci

$$\Sigma^i_k(W) := \left\{\rho : \pi_1(X, x) \longrightarrow \mathbb{C}^* \mid \dim(H^i(X, V_\rho \otimes W)) \geq k\right\} \tag{11}$$

passing through $\mathbf{1}$ *are sub-tori.*

Budur-Wang 2020: case X algebraic, algebraic methods, local systems *of geometric origin*.