

Lecture 1

I. Yoneda lemma

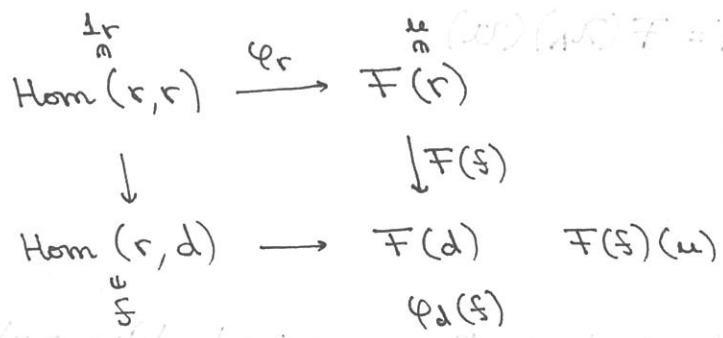
$\mathcal{C}$ : category  
 $r \in \mathcal{C}$

$F$ : functor  $\mathcal{C} \rightarrow \text{Set}$ , then there is a bijection

$$\text{Hom}(\text{Hom}(\mathbb{N}, -), F) \cong F(\mathbb{N})$$

Proof: Given  $\varphi$ ,  $\varphi_d: \text{Hom}(\mathbb{N}, d) \rightarrow F(d)$

$$\varphi_r(1_r) \in F(r)$$



Given  $u \in F(r)$ ,  $\varphi_d(f: r \rightarrow d) = F(f)(u)$

Def:  $F$  is representable if  $F \cong \text{Hom}(r, -)$

In this case:  $r$  is unique up to unique isom.

$$F \cong \text{Hom}(r, -) \cong \text{Hom}(r', -) \implies \varphi_r(1_r) \in \text{Hom}(r', r)$$

$u \in F(r)$  determines uniquely the isomorphism

$u$ : universal element

$F$  is representable by  $r$ ,  $u \in F(r) \iff \forall d \in \mathcal{C}, \forall x \in F(d) \exists ! f: r \rightarrow d \text{ st } x = F(f)(u)$

Example 1)  $P_1, \dots, P_r \in k[x_1, \dots, x_n]$

$F: k\text{-Alg} \rightarrow \text{Set}$

$$A \mapsto \{ (a_1, \dots, a_n) \in A^n / P_1(a_1, \dots, a_n) = \dots = P_r(a_1, \dots, a_n) = 0 \}$$

$\Rightarrow F$  is representable by  $R = k[x_1, \dots, x_n] / (P_1, \dots, P_r)$

$U = (x_1, \dots, x_n) \in \mathbb{R}^n, U \in F(R)$

$\forall \varphi \in F(A) \exists! f: R \rightarrow A$  st  $\varphi = F(f)(U)$

2)  $\mathcal{C} = \text{Set}, F(E) = \{ \text{subsets of } E \}$  y  $f: X \rightarrow Y, F(f) = f^{-1}$

$F$  is representable by  $\chi$ : characteristic map

$A \subseteq E \iff \chi_A: E \rightarrow \{0,1\}, A = \chi_A^{-1}(\{1\})$

Here  $R = \{0,1\}, U = \{1\} \in F(R)$

$\forall A \subseteq E, \exists! \chi_A: E \rightarrow R, A = \chi_A^{-1}(\{1\})$

$A \in F(E) \iff A = F(\chi_A)(U)$

3) If  $\mathcal{C}$  is any category,  $a, b \in \mathcal{C}$

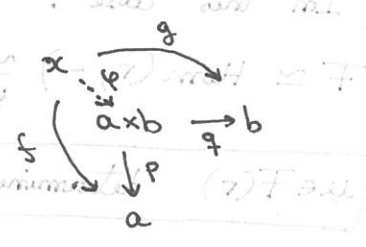
$F: \mathcal{C} \rightarrow \text{Set}$

$x \mapsto \text{Hom}(x, a) \times \text{Hom}(x, b)$

Exercise:  $F$  is representable  $\iff \exists$  (product in  $\mathcal{C}$  of  $a$  and  $b$ )  $:= a \times b$ .

Universal element:  $(p, q) \in F(a \times b), p \in \text{Hom}(a \times b, a), q \in \text{Hom}(a \times b, b)$   
 "canonical projections"

$\forall x \in \mathcal{C}, \forall (f, g) \in \text{Hom}(x, a) \times \text{Hom}(x, b) \exists! \varphi: x \rightarrow a \times b$   
 st  $(f, g) = F(\varphi)(p, q) = (p \circ \varphi, q \circ \varphi)$



4)  $\exists$  topological space  $BGL(r, \mathbb{R})$  and vector bundle  $E \rightarrow X$  of rank  $r$

$UGL(r, \mathbb{R}) \rightarrow BGL(r, \mathbb{R})$

st  $\forall$  manifold  $X$  and  $E \rightarrow X$  v.b. of rank  $r$

$\exists \varphi: X \rightarrow BGL(r, \mathbb{R})$  st  $E = \varphi^*(UGL(r, \mathbb{R}))$

$\varphi$  is unique up to homotopy

$B$ : classifying space

$U$ : universal bundle.

In this case,  $F : X \mapsto \left\{ \begin{array}{l} \text{vector bundles of} \\ \text{rank } r \text{ on } X \end{array} \right\}$   $F(\varphi : X \rightarrow Y) = \text{pull back.}$   
 $\Rightarrow F$  is representable by  $B$  and  $U$ .

5)  $r=1$  : line bundles.  
 Restrict to line bundles globally generated by  $N$  sections.  
 $(\exists s_1, \dots, s_N \text{ st } \forall x \in X \exists i s_i(x) \neq 0)$   
 then  $B = \mathbb{P}(\mathbb{R}^N)$  and  $U = \text{tautological line bundle.}$

$X \rightarrow B$   
 $x \mapsto \text{sections of } L \text{ vanishing at } x.$

More generally, for any  $G$ -topological group  $\exists BG$  and  $EG \rightarrow BG$   
 principal bundle st  $\forall X, \forall P \rightarrow X$  principal bundle  $\exists! \varphi : X \rightarrow BG$   
 st  $P = \varphi^*(EG).$

II. Principal bundles

Recall: A fiber bundle  $E \rightarrow X$  with fiber  $F$  and structure group  $G \curvearrowright F$   
 $\iff (U_i)$  cover of  $X$ . st  $E|_{U_i} \cong U_i \times F$  and  
 on  $U_i \cap U_j, \exists \varphi_{ij} : U_{ij} \rightarrow G$  st  $\varphi_{ij} \cdot \varphi_i^{-1}(x, f) = (x, \varphi_{ij}(x) \cdot f)$

Def: Principal bundle of group  $G \iff$  Fiber bundle with fiber  $F = G$ ,  
 $G \curvearrowright F = G$  by left translation

Remark: From  $(E, X, G, F)$  get a principal bundle by "forgetting" the  
 fiber :  $f \in F \rightarrow g \in G.$

From  $P$  principal bundle and  $F$  st  $G \curvearrowright F$   
 we get a fiber bundle with fiber  $F$

Example: Vector bundles of rank  $r \leftrightarrow$  Principal  $GL(r, \mathbb{R})$  bundles  
 $GL(r, \mathbb{R}) \cong \mathbb{R}^r$

But, if  $P$  is a principal bundle with structure group  $G$

Then,  $\exists$  global right  $G$ -action on  $P$  which preserves the fibers.

and  $P/G = X$

$P$  is trivial  $\iff$  has a section.

(If  $s: X \rightarrow P$  section then  $X \times G \rightarrow P$  homeomorphism)  
 $(x, g) \mapsto s(x) \cdot g$

Example of moduli problem

$X$  scheme, quasi-proj.  $M_X^r: \text{Sch}/k \rightarrow \text{Set}$

$M_X^r: \mathcal{U} \mapsto \{ \text{vector bundles of rank } r \text{ on } X \times \mathcal{U} \} / \text{isom}$

$E \in M_X^r(\mathcal{U}) \iff$  family of vector bundles parametrized by  $\mathcal{U}$ .

On  $f: \mathcal{U}' \rightarrow \mathcal{U}$ ,  $M_X^r(f):$  pull back along  $\text{id} \times f$ .

If  $M_X^r$  were representable by  $M_X^r \in \text{Sch}/k$  we would call it

a moduli space for vector bundles on  $X$ .

$\Rightarrow \text{Hom}(\text{Spec}(k), M_X^r) =$  vector bundles over  $X$ .

$1_M \in \text{Hom}(M_X^r, M_M^r) \iff$  universal family  $\mathcal{E}_{\text{univ}}$  on  $X \times M_X^r$

Problem: Usually  $M_X^r$  is not representable.

If  $\mathcal{U}$  scheme /  $k$ ,  $\mathcal{L}$  non-trivial line bundle over  $\mathcal{U}$ ,  $\text{pr}_\mathcal{U}: X \times \mathcal{U} \rightarrow \mathcal{U}$

If  $E \in M_X^r(\mathcal{U})$  and  $E' = E \otimes \text{pr}_\mathcal{U}^*(\mathcal{L})$

$E$  and  $E'$  are not isomorphic

$E, E'$  cover to  $\mathcal{U} \xrightarrow{\varphi} M_X^r$  and  $\mathcal{U} \xrightarrow{\varphi'} M_X^r$

Cover  $\mathcal{U}$  by  $(\mathcal{U}_i)$  st  $\mathcal{L}|_{\mathcal{U}_i}$  is trivial

then  $E|_{X \times \mathcal{U}_i} \cong E'|_{X \times \mathcal{U}_i} \Rightarrow \varphi|_{\mathcal{U}_i} = \varphi'|_{\mathcal{U}_i} \Rightarrow \varphi = \varphi'$

$\iff E$  and  $E'$  are isomorphic, contradiction!

2 solutions

Weakly dy of moduli spaces:

→ Restrict the v.b that are considered: semi-stability, GIT, etc.

→ Enlarge the category of schemes.

We will define the moduli stacks:

"functor"  $Sch/k \rightarrow Gpd$   
↖ groupoids.

$U \mapsto$  Vector bundles on  $X \times U$ .

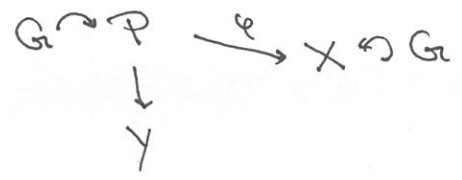
$f: U' \rightarrow U \mapsto$  "pull back along  $id \times f$ "

Remark!  $(g \circ f)^* \cong g^* \circ f^*$  (not equal, but isomorphic)

Quotient stack:  ~~$Sch/k$~~   $X \curvearrowright G$

$[X/G]: Sch/k \rightarrow Gpd$

$Y \mapsto \{ \mathcal{P} \text{ principal bundle on } Y + \varphi: \mathcal{P} \rightarrow X \}$   
 $G$ -equiv. morphism



Idea: Map  $\mathcal{P} \xrightarrow{\varphi} X$   $G$ -equiv  $\Rightarrow \mathcal{P}/G = Y \rightarrow "X/G"$ .

Remark:  $G \curvearrowright *$  maps  $Y \rightarrow [* / G] \leftrightarrow \mathcal{P}$  principal bundle on  $Y$

$\Rightarrow [* / G] = BG$ .