# Stacky Seminar

## Lecture 1

### I. Yoneda lemma

$\mathcal{C}$ : category

$r \in \mathcal{C}$

$F$ : functor $\mathcal{C} \to \mathrm{Set}$ , then there is a bijection

$$\mathrm{Hom}(\mathrm{Hom}(r, -), F) \simeq F(r)$$

Proof : Given $\varphi \in$ , $\varphi_d : \mathrm{Hom}(r, d) \to F(d)$

$$\varphi_r(1_r) \in F(r)$$

$$\begin{tikzcd}
\overset{1_r}{\overset{\cap}{\mathrm{Hom}}}(r,r) \arrow[r, "\varphi_r"] \arrow[d] & \overset{u}{\overset{\cap}{F}}(r) \arrow[d, "F(f)"] \\
\underset{\overset{\in}{f}}{\mathrm{Hom}(r,d)} \arrow[r] & \underset{\varphi_d(f)}{F(d)} \quad F(f)(u)
\end{tikzcd}$$

Given $u \in F(r)$ , $\varphi_d(f : r \to d) = F(f)(u)$ . ∎

Def : $F$ is representable if $F \simeq \mathrm{Hom}(r, -)$

In this case : $r$ is unique up to unique isom.

$F \simeq \mathrm{Hom}(r, -) \underset{\varphi}{\simeq} \mathrm{Hom}(r', -) \Rightarrow \varphi_r(1_r) \in \mathrm{Hom}(r', r)$

$\boxed{u \in F(r) \text{ determines uniquely the isomorphism}}$

$u$ : universal element

$F$ is representable by $r$, $u \in F(r) \iff \forall d \in \mathcal{C}, \forall x \in F(d)$

$\exists ! f : r \to d$ st $x = F(f)(u)$.

Example 1) $P_1, \dots, P_r \in k[x_1, \dots, x_n]$

$F : k\text{-Alg} \to \mathrm{Set}$

$$A \longmapsto \{(a_1, \dots, a_n) \in A^n \,/\, P_1(a_1, \dots, a_n) = \dots = P_r(a_1, \dots, a_n) = 0\}$$

$\Rightarrow \mathcal{F}$ is representable by $R = k[x_1, \dots, x_n]/(P_1, \dots, P_r)$

$\mathcal{U} = (x_1, \dots, x_n) \in R^n \quad , \quad \mathcal{U} \in \mathcal{F}(R)$

$\forall p \in \mathcal{F}(A) \quad \exists!\, f : R \to A \quad$ st $\quad p = \mathcal{F}(f)(\mathcal{U})$

2) $\mathcal{C} = \mathrm{Set} \quad , \quad \mathcal{F}(E) = \{\text{subsets of } E\} \quad$ if $f : X \to Y \ , \ \mathcal{F}(f) = f^{-1}$

$\mathcal{F}$ is representable by $\chi$ : characteristic map

$A \subseteq E \iff \chi_A : E \to \{0,1\} \quad , \quad A = \chi_A^{-1}(\{1\})$

Here $R = \{0,1\} \quad , \quad \mathcal{U} = \{1\} \in \mathcal{F}(R)$

$\forall A \subseteq E \ , \ \exists!\, \chi_A : E \to R \quad , \quad A = \chi_A^{-1}(\{1\})$

$A \in \mathcal{F}(E) \qquad\qquad\qquad A = \mathcal{F}(\chi_A)(\mathcal{U})$

3) If $\mathcal{C}$ is any category , $a, b \in \mathcal{C}$

$\mathcal{F} : \mathcal{C} \to \mathrm{Set}$

$\quad x \mapsto \mathrm{Hom}(x,a) \times \mathrm{Hom}(x,b)$

Exercise: $\mathcal{F}$ is representable $\iff \exists$(product in $\mathcal{C}$ of $a$ and $b$) $:= a \times b$.

Universal element: $(p, q) \in \mathcal{F}(a \times b) \ , \ p \in \mathrm{Hom}(a \times b, a)$ "canonical projections"

$q \in \mathrm{Hom}(a \times b, b)$

$\forall x \in \mathcal{C} \ , \ \forall (f,g) \in \mathrm{Hom}(x,a) \times \mathrm{Hom}(x,b) \quad \exists!\ x \xrightarrow{\varphi} a \times b$

st $(f,g) = \mathcal{F}(\varphi)(p,q) = (p \circ \varphi, q \circ \varphi)$

$$\begin{tikzcd} x \arrow[d, dashed, "\varphi"] \arrow[dr, bend left, "g"] \arrow[dd, bend right, "f"'] & \\ a \times b \arrow[r, "q"] \arrow[d, "p"] & b \\ a & \end{tikzcd}$$

4) $\exists$ topological space $BGL(r, \mathbb{R})$ and vector bundle
$\forall r \geq 1$ of rank $r$

$\mathcal{U}GL(r,\mathbb{R}) \to BGL(r,\mathbb{R})$

st $\forall$ manifold $X$ and for all $E \to X$ v.b. of rank $r$

$\exists\ \varphi : X \to BGL(r,\mathbb{R}) \quad$ st $\quad E = \varphi^*(\mathcal{U}GL(r,\mathbb{R}))$

$\varphi$ is unique up to homotopy

$B$ : classifying space

$\mathcal{U}$ : universal bundle.

In this case,

$$\mathcal{F} : X \longmapsto \left\{ \begin{array}{c} \text{vector bundles of} \\ \text{rank } r \text{ on } X \end{array} \right\} \Big/ \text{isom} \quad , \quad \mathcal{F}(\varphi : X \to Y) = \text{pullback} .$$

$\Rightarrow \mathcal{F}$ is representable by $B$ and $\mathcal{U}$.

5) $r = 1$ : Line bundles.
Restrict to line bundles globally generated by $N$ sections.
($\exists s_1, \dots, s_N$ st $\forall x \in X$ $\exists i$ $s_i(x) \neq 0$)
then $B = \mathbb{P}(\mathbb{R}^N)$ and $\mathcal{U}$ = tautological line bundle.

$X \longrightarrow B$
$x \longmapsto$ sections of $L$ vanishing at $x$.

More generally, for any $G$ topological group $\exists BG$ and $EG \to BG$ principal bundle st $\forall X$, $\forall P \to X$ principal bundle $\exists ! \, \varphi : X \to BG$ st $P = \varphi^*(EG)$.

II. Principal bundles

Recall: A fiber bundle $E \to X$ over $X$ with fiber $F$ and structure group $G \curvearrowright F$
$\longleftrightarrow$ $(\mathcal{U}_i)$ cover of $X$. st $E|_{\mathcal{U}_i} \underset{\psi_i}{\overset{\sim}{\to}} \mathcal{U}_i \times F$ and

On $\mathcal{U}_i \cap \mathcal{U}_j$, $\exists \, \varphi_{ij} : \mathcal{U}_{ij} \to G$ st $\psi_j \circ \psi_i^{-1}(x, \underset{F}{\underset{\cap}{f}}) = (x, \varphi_{ij}(x) \cdot f)$

Def: Principal bundle of group $G \longleftrightarrow$ Fiber bundle with fiber $F = G$, $G \curvearrowright F = G$ by left translation.

Rmk: From $(E, X, G, F)$ get a principal bundle by "forgetting" the fiber : $f \in F \longrightarrow g \in G$.

From $P$ principal bundle and $F$ st $G \curvearrowright F$
we get a fiber bundle with fiber $F$

Example: Vector bundles of rank $r$ $\longleftrightarrow$ Principal $GL(r, \mathbb{R})$ bundles
$GL(r, \mathbb{R}) \curvearrowright \mathbb{R}^r$

But, if $P$ is a principal bundle with structure group $G$

Then, $\exists$ global right $G$-action on $P$ which preserves the fibers.

and $P/G = X$

$P$ is trivial $\iff$ has a section.

(If $s: X \to P$ section then $X \times G \to P$ homeomorphism)
$(x, g) \mapsto s(x) \cdot g$

Example of moduli problem

$X$ scheme, quasi-proj. $\mathcal{M}_X^r : Sch/k \to Set$

$\mathcal{M}_X^r : \mathcal{U} \mapsto \left\{ \begin{array}{c} \text{vector bundles of rank } r \\ \text{on } X \times \mathcal{U} \end{array} \right\} / \text{isom}$.

$\mathcal{E} \in \mathcal{M}_X^r(\mathcal{U}) \longleftrightarrow$ family of vector bundles parametrized by $\mathcal{U}$.

On $f: \mathcal{U}' \to \mathcal{U}$, $\mathcal{M}_X^r(f)$ : pull back along $id \times f$.

If $\mathcal{M}_X^r$ where representable by $M_X^r \in Sch/k$ we would call it a moduli space for vector bundles on $X$.

$\Rightarrow Hom(Spec(k), M_X^r) =$ vector bundles over $X$.

$1_M \in Hom(M_X^r, M_M^r) \longleftrightarrow$ universal family $\mathcal{E}_{univ}$ on $X \times M_X^r$

Problem: Usually $\mathcal{M}_X^r$ is not representable.

If $\mathcal{U}$ scheme $/k$, $\mathcal{L}$ non-trivial line bundle over $\mathcal{U}$, $pr_{\mathcal{U}}: X \times \mathcal{U} \to \mathcal{U}$

If $\mathcal{E} \in \mathcal{M}_X^r(\mathcal{U})$ and $\mathcal{E}' = \mathcal{E} \otimes pr_{\mathcal{U}}^*(\mathcal{L})$

$\mathcal{E}$ and $\mathcal{E}'$ are not isomorphic

$\mathcal{E}, \mathcal{E}'$ corresp to $\mathcal{U} \xrightarrow{\varphi} M_X^r$ and $\mathcal{U} \xrightarrow{\varphi'} M_X^r$

Cover $\mathcal{U}$ by $(\mathcal{U}_i)$ st $\mathcal{L}|_{\mathcal{U}_i}$ is trivial

then $\mathcal{E}|_{X \times \mathcal{U}_i} \simeq \mathcal{E}'|_{X \times \mathcal{U}_i} \Rightarrow \varphi|_{\mathcal{U}_i} = \varphi'|_{\mathcal{U}_i} \Rightarrow \varphi = \varphi'$

$\Leftrightarrow \mathcal{E}$ and $\mathcal{E}'$ are isomorphic, contradiction!

2 solutions

- Weaker def of moduli spaces:
  - Restrict the v.b that are considered: semi-stability, GIT, etc.

→ Enlarge the category of schemes.

We will define the <u>moduli stack</u>:

"functor" $Sch/k \longrightarrow Gpd$ (groupoids.)

$$\mathcal{U} \longmapsto \text{Vector bundles on } X \times \mathcal{U}.$$

$f : \mathcal{U}' \to \mathcal{U} \mapsto$ "pull back along $id \times f$"

<u>Rmk</u>! $(g \circ f)^* \simeq g^* \circ f^*$ (not <u>equal</u>, but isomorphic)

<u>Quotient stack</u>: $X \circlearrowleft G$

$[X/G] : Sch/k \longrightarrow Gpd$

$$Y \longmapsto \{P \text{ principal bundles on } Y + \varphi : P \to X\}$$

G-equiv. morphism

$$\begin{array}{ccc} G \curvearrowright P & \xrightarrow{\varphi} & X \circlearrowleft G \\ \downarrow & & \\ Y & & \end{array}$$

<u>Idea</u>: Map $P \xrightarrow{\varphi} X$ G-equiv $\Rightarrow P/G = Y \longrightarrow$ "$X/G$".

<u>Rmk</u>: $G \curvearrowright *$ maps $Y \to [*/G] \longleftrightarrow P$ principal bundle on $Y$

$\Rightarrow [*/G] = BG$.