

Lecture 1

I. Yoneda Lemma

\mathcal{C} : category $\mathcal{D} = \mathcal{C}^{\text{op}}$, $\mathcal{X} = \mathcal{C}$ $\mathcal{Y} = \mathcal{D}$ \mathcal{F} \mathcal{G} functors $\mathcal{C} \rightarrow \mathcal{D}$ $\mathcal{F} \circ \mathcal{G} = \mathcal{I}$ (identity)

\mathcal{F} : functor $\mathcal{C} \rightarrow \text{Set}$, then there is a bijection $\text{Hom}(\mathcal{F}(A), \mathcal{F}(B)) \cong \text{Hom}(A, \mathcal{G}(B))$ \mathcal{F} gel Schauspieler in \mathcal{F}

$$\text{Hom}(\text{Hom}(A, -), \mathcal{F}) \cong \mathcal{F}(A)$$

Proof: Given φ , $\varphi_d: \text{Hom}(A, d) \rightarrow \mathcal{F}(d)$ $\mathcal{F} \circ \mathcal{G} = \mathcal{I}$ $\mathcal{G} \circ \mathcal{F} = \mathcal{I}$

$$\varphi_r(\tau) \in \mathcal{F}(r)$$

$$\begin{array}{ccc} \text{Hom}(r, r) & \xrightarrow{\varphi_r} & \mathcal{F}(r) \\ \downarrow & & \downarrow \mathcal{F}(s) \\ \text{Hom}(r, d) & \longrightarrow & \mathcal{F}(d) \quad \mathcal{F}(s)(u) \end{array} \quad \begin{array}{l} \text{So } \varphi_r \text{ is a map from } \mathcal{F}(r) \text{ to } \mathcal{F}(d) \\ \text{and } \mathcal{F}(s) \text{ is a map from } \mathcal{F}(r) \text{ to } \mathcal{F}(d) \end{array}$$

$$\begin{array}{ccc} \text{Hom}(r, d) & \longrightarrow & \mathcal{F}(d) \quad \mathcal{F}(s)(u) \\ \downarrow \varphi_d(s) & & \downarrow \mathcal{F}(s) \\ \text{Hom}(r, s) & \longrightarrow & \mathcal{F}(s) \end{array} \quad \begin{array}{l} \text{and } \varphi_d(s) \text{ is a map from } \mathcal{F}(r) \text{ to } \mathcal{F}(s) \\ \text{and } \mathcal{F}(s) \text{ is a map from } \mathcal{F}(r) \text{ to } \mathcal{F}(s) \end{array}$$

Given $\varphi_r \in \mathcal{F}(r)$, $\varphi_d(s: r \rightarrow d) = \mathcal{F}(s)(u)$. $\mathcal{F} \circ \mathcal{G} = \mathcal{I}$ $\mathcal{G} \circ \mathcal{F} = \mathcal{I}$

Def: \mathcal{F} is representable by r if $\mathcal{F} \cong \text{Hom}(r, -)$

In this case: \mathcal{F} is unique up to isomorphism.

$\mathcal{F} \cong \text{Hom}(r, -) \cong \text{Hom}(r', -) \Rightarrow \varphi_{r'}(r) \in \text{Hom}(r, r')$ $\mathcal{F} \circ \mathcal{G} = \mathcal{I}$ $\mathcal{G} \circ \mathcal{F} = \mathcal{I}$

$\varphi_r(r) \in \mathcal{F}(r)$ determines uniquely the isomorphism

\mathcal{F} is representable by r , $\varphi_r(r) \in \mathcal{F}(r) \Leftrightarrow \forall d \in \mathcal{D}, \forall s \in \mathcal{F}(d)$

$\exists! f: r \rightarrow d$ s.t. $\varphi_d(f) = \mathcal{F}(s)(u)$

Example 1: $P_1, \dots, P_n \in \mathbb{K}[x_1, \dots, x_m]$ $(\mathbb{R}, \mathbb{C}, \mathbb{W}) \xrightarrow{\varphi} \mathcal{F}$ to $(\mathbb{R}, \mathbb{C}, \mathbb{W}) \xrightarrow{\varphi} \mathcal{F}$

$\mathcal{F}: \text{la-Alg} \rightarrow \text{Set}$ \mathcal{F} representable at φ algebra in \mathcal{F}

$A \mapsto \{(a_1, \dots, a_m) \in A^m / P_1(a_1, \dots, a_m) = \dots = P_n(a_1, \dots, a_m) = 0\}$ \mathcal{F} universal locoalgebra in \mathcal{F}

$\Rightarrow \mathcal{F}$ is representable by $R = k[x_1, \dots, x_m]/(P_1, \dots, P_r)$

$u = (x_1, \dots, x_m) \in R^m$, $u \in \mathcal{F}(R)$

Lemma

$\forall p \in \mathcal{F}(A)$ $\exists! f : R \rightarrow A$ st $p = \mathcal{F}(f)(u)$

commutative diagram

2) $\mathcal{C} = \text{Set}$, $\mathcal{F}(E) = \{\text{subsets of } E\}$ if $f : X \rightarrow Y$, $\mathcal{F}(f) = f^{-1}$ (for E satisfying \mathcal{F})

\mathcal{F} is representable by X : characteristic map with $\text{Set} \hookrightarrow \mathcal{F}$ satisfying \mathcal{F}

$A \subseteq E \Leftrightarrow X_A : E \rightarrow \{0, 1\}$, $A = X_A^{-1}(\{1\})$ (with $\{1\}$ with \mathcal{F})

Here $R = \{0, 1\}$, $u = \{1\} \in \mathcal{F}(R)$ (with \mathcal{F}) if $u \in \mathcal{F}$

$\forall A \subseteq E$, $\exists! X_A : E \rightarrow R$, $A = X_A^{-1}(\{1\})$

$A \in \mathcal{F}(E)$

$A = \mathcal{F}(X_A)(u)$

(a) $\mathcal{F} \hookrightarrow (0, 1)$ with \mathcal{F}

3) If \mathcal{C} is any category, $a, b \in \mathcal{C}$

$\mathcal{F} : \mathcal{C} \rightarrow \text{Set}$

$x \mapsto \text{Hom}(x, a) \times \text{Hom}(x, b)$

(a) $\mathcal{F} \hookrightarrow (0, 1)$ with \mathcal{F}

(b) $\mathcal{F} \hookrightarrow (0, 1)$ with \mathcal{F}

Exercise: \mathcal{F} is representable $\Leftrightarrow \exists$ (product in \mathcal{C} of a and b): $a \times b$.

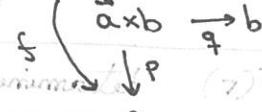
Universal element: $(p, q) \in \mathcal{F}(a \times b)$, $p \in \text{Hom}(a \times b, a)$

"canonical" "projections"

$q \in \text{Hom}(a \times b, b)$ unique w.r.t \mathcal{F} : $\exists!$

$\forall x \in \mathcal{C}$, $\forall (f, g) \in \text{Hom}(x, a) \times \text{Hom}(x, b)$ $\exists!$ $\xrightarrow{x} a \times b$: $x \mapsto (f, g)$ with \mathcal{F}

st $(f, g) = \mathcal{F}(\varphi)(p, q) = (p \circ \varphi, q \circ \varphi)$ ($\varphi : x \rightarrow a \times b$ with \mathcal{F})



4) \exists topological space $BGL(r, \mathbb{R})$ and vector bundle E unique to \mathcal{F} (with \mathcal{F})

$\forall r \geq 1$ of rank r

bundle isomorphism: \sim

$UGL(r, \mathbb{R}) \rightarrow BGL(r, \mathbb{R})$

st $(a) \mathcal{F}$ is a manifold X and \sim for all $E \rightarrow X$ v.b. of rank r w.r.t \mathcal{F}

$\exists \varphi : X \rightarrow BGL(r, \mathbb{R})$ st $E = \varphi^*(UGL(r, \mathbb{R}))$

φ is unique up to homotopy

for \mathcal{F} is full and \mathcal{F}

B : classifying space

U : universal bundle.

In this case, local (B,D,B) definition of a fiber bundle satisfies: (2)
 $F : X \mapsto \{ \text{vector bundles of rank } r \text{ on } X \}$, $F(\varphi : X \rightarrow Y) = \text{pull-back}.$

So group actions from fibered category is at P if, that
 $\Rightarrow F$ is representable by B and U : no matter B before looking at F (next)

5) $r=1$: Line bundles.

Restrict to line bundles globally generated by N sections.
 $(\exists s_1, \dots, s_N \text{ st } \forall x \in X \exists i s_i(x) \neq 0)$

then $B = \mathbb{P}(R^N)$ and $U = \text{tautological line bundle}.$

$$X \rightarrow B$$

$x \mapsto$ sections of L vanishing at x .

and every fibration is representable

More generally, for any G -topological group $\exists B G$ and $E G \rightarrow B G$
 principal bundle $\exists X \xrightarrow{\varphi} P \rightarrow X$ principal bundle $\exists ! \varphi : X \rightarrow B G$
 $\text{st } P = \varphi^*(E G).$

II. Principal bundles

$$E \xrightarrow{\pi} X$$

Recall: A fiber bundle over X with fiber F and structure group G
 $\leftrightarrow (U_i)$ cover of X . st $E|_{U_i} \xrightarrow{\pi_i} U_i \times F$ and each fiber

on $U_i \cap U_j$, $\exists \varphi_{ij} : U_{ij} \rightarrow G$ st $\varphi_{ij} \circ \pi_j \circ \varphi_{ij}^{-1}(x, f) = (x, \varphi_{ij}(x) \cdot f)$

Deg: Principal bundle of group G \leftrightarrow Fiber bundle with fiber $F = G$,

$G \curvearrowright F = G$ by left translation

Rank: From (E, X, G, F) get a principal bundle by "forgetting the"

given: $f \in F \rightarrow g \in G$. (6) 3.4.3 = 3 $\text{and } (6) 3.4.3 = 3$

From P principal bundle and F st $G \curvearrowright F$ $\text{for } G = B$ has 3
 we get a fiber bundle with fiber F $\text{for } G = B$ has 3 at grade 3, 3

induced in pull-back (6) for G has 3

$\varphi = \varphi \circ \pi$ with φ with G has 3 $\text{and } B$ with G has 3 with

\mathbb{P} multiplication $\text{induced on } G$ has 3 or,

Example: Vector bundles of rank $r \leftrightarrow$ Principal $GL(r, \mathbb{R})$ bundles and vice versa.
 $GL(r, \mathbb{R}) \cong \mathbb{R}^r \times GL(r, \mathbb{R})$

But, if P is a principal bundle with structure group G

Then, \exists global right G -action on P which preserves the fibers. If so,
and $P/G = X$

P is trivial \Leftrightarrow has a section. (i.e. \exists global right G -action on P such that $s(x) \cdot g = s(x)$ for all $x \in X$ and $g \in G$)
(Iff $s: X \rightarrow P$ section then $X \times G \xrightarrow{\text{right action}} P$ is homeomorphism)

Example of moduli problem

X scheme, quasi-proj. $M_X^r: Sch/k \rightarrow$ Set

$M_X^r: U \mapsto \{ \text{vector bundles of rank } r \}$ from U via pullback along $\pi_U: X \times U \rightarrow U$

$E \in M_X^r(U) \leftrightarrow$ family of vector bundles parametrized by local G -bundles

On $f: U' \rightarrow U$, $M_X^r(f)$: pull back along $\text{id} \times f$.

If M_X^r where representable by $F \in Sch/k$ we would call it a moduli space for vector bundles on X .

$\Rightarrow \text{Hom}(\text{Spec}(k), M_X^r) =$ vector bundles over X .

$\lambda_M^r \in \text{Hom}(M_X^r, M_M^r) \leftrightarrow$ universal family E_{univ} on $X \times M_X^r$

Problem: Usually M_X^r is not representable.

If U scheme/k, \mathcal{L} non-trivial line bundle over U , $\text{pr}_U^*: X \times U \rightarrow U$

If $E \in M_X^r(U)$ and $E' = E \otimes \text{pr}_U^*(\mathcal{L})$

E and E' are not isomorphic to \mathcal{F} has global sections \mathcal{F} and \mathcal{F}'

E, E' cover U by $U_i \xrightarrow{\varphi_i} M_X^r$ and $U_i \xrightarrow{\varphi'_i} M_X^r$ global sections φ_i, φ'_i

Cover U by (U_i) st $\mathcal{L}|_{U_i}$ is trivial

then $E|_{X \times U_i} \cong E'|_{X \times U_i} \Rightarrow \varphi|_{U_i} = \varphi'|_{U_i} \Rightarrow \varphi = \varphi'$

$\Leftrightarrow E$ and E' are isomorphic, contradiction!

2 solutions

↳ Weaken dg of moduli spaces:

↳ Restrict the v.b. that are considered: semi stability, GIT, etc.

→ Enlarge the category of schemes.

We will define the moduli stack:

"functor" $\text{Sch}/\mathbb{A} \rightarrow \text{Gpd}$
 \sim groupoids.

$U \mapsto$ Vector bundles on $X \times U$.

$f: U \rightarrow U \mapsto$ "pull back along $\text{id} \times f$ "

Remark! $(g \circ f)^* \cong g^* \circ f^*$ (not equal, but isomorphic)

Quotient stack: ~~X/G~~ $\sim X/G$

$[X/G]: \text{Sch}/\mathbb{A} \rightarrow \text{Gpd}$

$Y \mapsto \left\{ \begin{array}{l} P \text{ principal bundle on } Y + \varphi: P \rightarrow X \\ G\text{-equiv. morphism} \end{array} \right\}$

$$\begin{array}{ccc} G \curvearrowright P & \xrightarrow{\varphi} & X \curvearrowright G \\ \downarrow & & \\ Y & & \end{array}$$

Idea: Map $P \xrightarrow{\varphi} X$ G -equiv $\Rightarrow P/G = Y \rightarrow "X/G"$.

Remark: $G \curvearrowright *$ maps $Y \rightarrow [*/G]$ $\leftrightarrow P$ principal bundle on Y
 $\Rightarrow [*/G] = BG$.