Lecture 2 :

If X scheme $/S$

Stack : "functor" $Sch \to Gpd$.

Moduli stack of vector bundles of rank r on X

$\mathcal{U} \in Sch/S \longmapsto$ vector bundles of rank r on $X \times \mathcal{U}$ + their isomorphisms

If $f : \mathcal{U}' \to \mathcal{U}$

$f^* : Bun^r_X(\mathcal{U}) \to Bun^r_X(\mathcal{U}')$ pull-back via $(id \times f)$

$(g \circ f)^* \cong f^* \circ g^*$

Quotient stack : $G \curvearrowright X$

$[X/G] : \mathcal{U} \longmapsto$ G-principal bundles P on $\mathcal{U}$
+ G-equiv. map $P \to X$
+ their isomorphisms

If $f : \mathcal{U}' \to \mathcal{U}$, $[X/G](\mathcal{U}) \to [X/G](\mathcal{U}')$ pullback of principal bundle.

I. Grothendieck topology

$\mathcal{C}$: category with fiber products

A Grothendieck topology on $\mathcal{C}$ is : For each $\mathcal{U} \in \mathcal{C}$ a set of families

$\tau(\mathcal{U})$: set of coverings. $\{\mathcal{U}_i \to \mathcal{U}\}$ st

1) If $\mathcal{U}' \to \mathcal{U}$ isomorphism, then $\{\mathcal{U}' \to \mathcal{U}\} \in \tau(\mathcal{U})$

2) (Transitivity) : If $\{\mathcal{U}_i \xrightarrow{\varphi_i} \mathcal{U}\} \in \tau(\mathcal{U})$ and $\forall i : \{\mathcal{U}_{ij} \xrightarrow{\varphi_{ij}} \mathcal{U}_i\} \in \tau(\mathcal{U}_i)$
then $\{\mathcal{U}_{ij} \xrightarrow{\varphi_i \circ \varphi_{ij}} \mathcal{U}\}_{i,j} \in \tau(\mathcal{U})$

3) (Base change) If $\{\mathcal{U}_i \to \mathcal{U}\} \in \tau(\mathcal{U})$ and $V \to \mathcal{U}$
then $\{V \times_{\mathcal{U}} \mathcal{U}_i \to V\} \in \tau(V)$.

$\mathcal{C}$ + Grothendieck top : Site.

Basic example: X topological space.

$\mathcal{O}(X)$: category of open sets
maps: $V \hookrightarrow \mathcal{U}$.

$\tau(\mathcal{U})$ = set of open covers of $\mathcal{U}$
= set of families $\{\mathcal{U}_i \xrightarrow{\varphi_i} \mathcal{U}\}$ st $\bigcup_i \varphi_i(\mathcal{U}_i) = \mathcal{U}$.

If X scheme/S we define:

Small Zariski site: Objects $\mathcal{U} \hookrightarrow X$
Morphisms

$$\begin{tikzcd} \mathcal{U}' \arrow[rr] \arrow[dr] & & \mathcal{U} \arrow[dl] \\ & X & \end{tikzcd}$$

Covering of $\mathcal{U}$: $\{\mathcal{U}_i \xrightarrow{\varphi_i} \mathcal{U}\}$ st $\bigcup_i \varphi_i(\mathcal{U}_i) = \mathcal{U}$.

Small étale site: Objects: Étale morphisms $\mathcal{U} \to X$, $\mathcal{U} \in Sch/S$
Morphisms: $\mathcal{U} \to \mathcal{U}'$ étale

$$\begin{tikzcd} \mathcal{U} \arrow[rr] \arrow[dr] & & \mathcal{U}' \arrow[dl] \\ & X & \end{tikzcd}$$

Covers of $\mathcal{U}$: $\{\mathcal{U}_i \xrightarrow{\varphi_i} \mathcal{U}\}$ φ_i: étale st $\bigcup_i \varphi_i(\mathcal{U}_i) = \mathcal{U}$.

Same construction with étale replaced by:
- smooth
- fppf
- fpqc

Remark:

Open embedding $\Rightarrow$ Étale $\Rightarrow$ Smooth $\Rightarrow$ fppf $\Rightarrow$ fpqc

Big sites • Étale: Objects: Sch/S
Morphisms: morphisms of schemes
Covers of X: same.

Def: If $\mathcal{C}$ is a site.
A presheaf on $\mathcal{C}$ is $\mathcal{F}: \mathcal{C}^{op} \to Set$
A sheaf on $\mathcal{C}$ is a presheaf $\mathcal{F}: \mathcal{C}^{op} \to Set$ such that
$\forall \mathcal{U} \in \mathcal{C}$, $f, g \in \mathcal{F}(\mathcal{U})$ if $\forall \{\mathcal{U}_i \xrightarrow{\varphi_i} \mathcal{U}\} \in \tau(\mathcal{U})$
$f|_{\mathcal{U}_i} = g|_{\mathcal{U}_i}$ (Here: $\mathcal{F}(\varphi_i)(f) = \mathcal{F}(\varphi_i)(g)$) $\Rightarrow f = g$.

2. $\forall \mathcal{U} \in \mathcal{C}$, $\forall \{\mathcal{U}_i \xrightarrow{\varphi_i} \mathcal{U}\} \in \tau(\mathcal{U})$

$\forall f_i \in \mathcal{F}(\mathcal{U}_i)$ st $\mathcal{F}(\varphi_{i,j,i})(f_i) = \mathcal{F}(\varphi_{i,j,j})(f_j)$ in $\mathcal{F}(\mathcal{U}_i \times_{\mathcal{U}} \mathcal{U}_j)$

"$\mathcal{U}_i \cap \mathcal{U}_j \subseteq \mathcal{U}_i$" "$\mathcal{U}_i \cap \mathcal{U}_j \subseteq \mathcal{U}_j$"

$[\varphi_{i,j,i} : \mathcal{U}_i \times_{\mathcal{U}} \mathcal{U}_j \to \mathcal{U}_i]$

$\Rightarrow \exists f \in \mathcal{F}(\mathcal{U})$ st $\forall i$ $f|_{\mathcal{U}_i} = f_i$.

Rmk: Sheaf for the fpqc top $\Rightarrow$ sheaf for fppf $\Rightarrow$ smooth $\Rightarrow$ étale $\Rightarrow$ Zar.

If $X \in Sch/S$ we can define $\mathrm{Hom}(-, X) : (Sch/S)^{op} \to Set$

Thm: This is a sheaf for the Zariski topology. (tautological)

Thm (Grothendieck): This is a sheaf for the fpqc topology ("descent theory").

Remark: Schemes $\subseteq$ Alg. spaces $\subseteq$ Sheaves $\subseteq$ Functors $Sch^{op}/_S \to Set$.

II. 2-categories

Cat = category of all categories

Objects: categories

Morphisms: functors.

+ between two functors $\mathcal{C} \underset{G}{\overset{F}{\rightrightarrows}} \mathcal{D}$ natural transformations.

+ vertical composition of natural transformations

$$\begin{tikzcd}
X \in \mathcal{C} \arrow[r, bend left=60, "F"] \arrow[r, "G" description] \arrow[r, bend right=60, "H"'] \arrow[r, phantom, shift left=3, "\Downarrow\alpha"] \arrow[r, phantom, shift right=3, "\Downarrow\beta"] & \ast\mathcal{D}
\end{tikzcd}$$

$\alpha_X : F(X) \to G(X)$

$\beta_X : G(X) \to H(X)$

$\beta \circ \alpha : F \Rightarrow H$

$(\beta \circ \alpha)_X : \beta_X \circ \alpha_X$

+ horizontal composition:

$$\begin{tikzcd}
X \in \mathcal{C} \arrow[r, bend left=50, "F"] \arrow[r, bend right=50, "F'"'] \arrow[r, phantom, "\alpha\Downarrow"] & \ast\mathcal{D} \arrow[r, bend left=50, "G"] \arrow[r, bend right=50, "G'"'] \arrow[r, phantom, "\beta\Downarrow"] & \ast\mathcal{E} \arrow[r, squiggly] & \mathcal{C} \arrow[r, bend left=50, "G\circ F"] \arrow[r, bend right=50, "G'\circ F'"'] \arrow[r, phantom, "\beta * \alpha \Downarrow"] & \mathcal{E}
\end{tikzcd}$$

$\beta * \alpha : G \circ F \Rightarrow G' \circ F'$

$X \in \mathcal{C}$: $\alpha_X : F(X) \to F'(X)$

$G\alpha_X : GF(X) \to GF'(X)$

$\beta_{F'(X)} : G(F'(X)) \to G'(F'(X))$

$\beta_{F'(X)} \circ G\alpha_X : GF(X) \to G'F'(X)$

Déf : A 2-category $\mathcal{C}$ is :

$\mathcal{C}_0$: Set of objects = Set of 0-morphisms.

$\mathcal{C}_1$: Set of 1-morphisms

$\mathcal{C}_2$: Set of 2-morphism

st $\mathcal{C}_0, \mathcal{C}_1$ categories.

+ between two parallel 1-morphisms $\begin{tikzcd} x \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] & y \end{tikzcd}$ there are 2-morphisms

$$\begin{tikzcd} x \arrow[r, bend left=50, "f"{name=U}] \arrow[r, bend right=50, "g"'{name=D}] \arrow[Rightarrow, from=U, to=D, "\alpha"'] & y \end{tikzcd}$$

+ identity 2-morphism

+ vertical composition of 2-morphisms, associativity, identity.

+ horizontal composition

$$\begin{tikzcd} x \arrow[r, bend left=50, "f"{name=U1}] \arrow[r, bend right=50, "f'"'{name=D1}] \arrow[Rightarrow, from=U1, to=D1, "\alpha"'] & y \arrow[r, bend left=50, "g"{name=U2}] \arrow[r, bend right=50, "g'"'{name=D2}] \arrow[Rightarrow, from=U2, to=D2, "\beta"'] & z \end{tikzcd} \qquad \beta * \alpha : g \circ f \Rightarrow g' \circ f'.$$

+ Axiom :

$$\begin{tikzcd} x \arrow[r, bend left=60] \arrow[r] \arrow[r, bend right=60] \arrow[r, phantom, shift left=3, "\alpha\Downarrow"] \arrow[r, phantom, shift right=3, "\beta\Downarrow"] & y \arrow[r, bend left=60] \arrow[r] \arrow[r, bend right=60] \arrow[r, phantom, shift left=3, "\gamma\Downarrow"] \arrow[r, phantom, shift right=3, "\delta\Downarrow"] & z \end{tikzcd} \qquad (\delta \circ \gamma) * (\beta \circ \alpha) = (\delta * \beta) \circ (\gamma * \alpha)$$

Examples :

- Cat
- Gpd : 1-morphisms are isomorphisms
- Grp (groups) : Here : 2-morphism between $f, g : G \to H$ $\leftrightarrow$ conjugation in H.
- $\mathcal{C}$: $\mathcal{C}_0$ = topological spaces
  $\mathcal{C}_1$ = continuous maps. $f : X \to Y$
  $\mathcal{C}_2$ = homotopy $\begin{tikzcd} X \arrow[r, bend left=50, "f"{name=U}] \arrow[r, bend right=50, "g"'{name=D}] \arrow[Rightarrow, from=U, to=D, "H"'] & Y \end{tikzcd}$ modulo homotopies of homotopies ($\to$ vertical composition is well def)

$H : X \times [0,1] \to Y$

Other point of view :

A 2-category $\Leftrightarrow$ Category $\mathcal{C}$ st $\mathrm{Hom}(x,y)$ is a category (with vertical comp. of 2-morph.)

+

$$\mathrm{Hom}(y,z) \times \mathrm{Hom}(x,y) \to \mathrm{Hom}(x,z)$$
$$(g, f) \longmapsto g \circ f$$

is a bifunctor ($\approx$ distributivity $\leftrightarrow$ horizontal composition)

Functor between 2-categories : 2-functors.

All this is called <u>strict</u> 2-categories.

<u>Weak versions</u> :

Motivation : • A 1-morphism $f : x \to y$ is called <u>invertible</u> if $\exists g : y \to x$ st $gf = id_x$, $fg = id_y$

• Same for invertible 2-morphism.

→ A 1-morphism $f : x \to y$ is called an <u>equivalence</u> if $\exists g : y \to x$ + invertible 2-morphisms $gf \xrightarrow{\sim} id_x$, $fg \xrightarrow{\sim} id_y$.

<u>Example</u> : Cat : Functor being an equivalence of categories

Top : Homotopy equivalence.

<u>Recall</u> : $F : \mathcal{C} \to \mathcal{D}$ is an equiv. of categories

$\iff \forall y \in \mathcal{D}\ \exists x \in \mathcal{C}$ st $F(x) \xrightarrow{\sim} y$

$+ F : Hom(a,b) \to Hom(F(a), F(b))$ is a bijection.

<u>Def</u> : A <u>pseudo-functor</u> (weak 2-functor) is $F : \mathcal{C} \to \mathcal{D}$ between 2-categories st :

$\forall x \in \mathcal{C}$, $\exists \alpha_x$ invertible 2-morphism with $F(id_x) \underset{\sim}{\xrightarrow{\alpha_x}} id_{F(x)}$

and $\forall f, g$ 1-morphisms $\exists \varepsilon_{g,f}$: invertible 2-morphism st :

$$F(g \circ f) \underset{\sim}{\xrightarrow{\varepsilon_{g,f}}} F(g) \circ F(f)$$

+F strictly associative on 2-morphisms

+ Axioms on α, ε.

——— o ———

Weak 2-category : 2-morphisms, associative composition

For 1-morphisms $h \circ (g \circ f) \underset{\substack{\uparrow \\ \text{invertible} \\ \text{2-morphism}}}{\simeq} (h \circ g) \circ f$.

III. Stacks

<u>Def</u>: Let $\mathcal{C}$ be any category. A <u>pre stack</u> is a pseudo-functor $\mathcal{C}^{op} \to Gpd$

Let $\mathcal{C}$ be a site, a <u>stack</u> is a pseudo-functor $\mathcal{X} : \mathcal{C} \to Gpd$

[Notation : $x \in \mathcal{X}(\mathcal{U})$ and $f : \mathcal{U}' \to \mathcal{U}$, $x|_{\mathcal{U}'} := \mathcal{X}(f)(x)$]

• Gluing of objects :

$\forall \mathcal{U} \in \mathcal{C}$, $\forall \{\mathcal{U}_i \to \mathcal{U}\} \in \tau(\mathcal{U})$, $\forall x_i \in \mathcal{X}(\mathcal{U}_i)$

+ morphisms $\varphi_{ij} : x_i|_{\mathcal{U}_{ij}} \to x_j|_{\mathcal{U}_{ij}}$ st $\varphi_{ij}|_{\mathcal{U}_{ijk}} \circ \varphi_{jk}|_{\mathcal{U}_{ijk}} = \varphi_{ik}|_{\mathcal{U}_{ijk}}$

$\Rightarrow \exists\, x \in \mathcal{X}(\mathcal{U})$ and $\varphi_j : x|_{\mathcal{U}_i} \xrightarrow{\sim} x_i$ in $\mathcal{X}(\mathcal{U}_i)$

st $\varphi_{ji} \circ \varphi_i|_{\mathcal{U}_{ij}} = \varphi_j|_{\mathcal{U}_{ij}}$

• Isom. form a sheaf:

$\forall x, y \in \mathcal{X}(\mathcal{U})$, $\varphi_i : x|_{\mathcal{U}_i} \to y|_{\mathcal{U}_i}$ st $\varphi_i|_{\mathcal{U}_{ij}} = \varphi_j|_{\mathcal{U}_{ij}}$

$\Rightarrow \exists\, \eta : x \to y$ st $\eta|_{\mathcal{U}_i} = \varphi_i$

and $\forall x, y \in \mathcal{X}(\mathcal{U})$ $\forall \varphi : x \to y$, $\psi : x \to y$

st $\varphi|_{\mathcal{U}_i} = \psi|_{\mathcal{U}_i} \Rightarrow \varphi = \psi$

$\mathcal{U} \mapsto \mathrm{Isom}_{\mathcal{U}}(x, y)$ is a sheaf.