

Lecture 2:

If X scheme / S

Stack: "functor" $Sch \rightarrow Gpd.$

Moduli stack of vector bundles of rank r on X

$U \in Sch/S \mapsto$ vector bundles of rank r on $X \times U$ + their isomorphisms

If $f: U' \rightarrow U$

$f^*: Bun_X^r(U) \rightarrow Bun_X^r(U')$ pull-back via $(id \times f)$

$(g \circ f)^* \cong f^* \circ g^*$

Quotient stack: $G \curvearrowright X$

$[X/G]: U \mapsto$ G -principal bundles P on U
+ G -equiv. map $P \rightarrow X$
+ their isomorphisms

If $f: U' \rightarrow U$, $[X/G](U) \rightarrow [X/G](U')$ pull back of principal bundle.

I. Grothendieck topology

\mathcal{C} : category with fiber products

A Grothendieck topology on \mathcal{C} is: For each $U \in \mathcal{C}$ a set of families

$\tau(U)$: set of coverings. $\{U_i \rightarrow U\}$ st

1) If $U' \rightarrow U$ isomorphism, then $\{U' \rightarrow U\} \in \tau(U)$

2) (Transitivity): If $\{U_i \xrightarrow{\phi_i} U\} \in \tau(U)$ and $\forall i: \{U_{ij} \xrightarrow{\phi_{ij}} U_i\} \in \tau(U_i)$
then $\{U_{ij} \xrightarrow{\phi_i \circ \phi_{ij}} U\}_{i,j} \in \tau(U)$

3) (Base change) If $\{U_i \rightarrow U\} \in \tau(U)$ and $V \rightarrow U$
then $\{V \times_U U_i \rightarrow V\} \in \tau(V)$.

\mathcal{C} + Grothendieck top: Site.

Basic example: X topological space.

$\mathcal{O}(X)$: category of open sets

maps: $V \hookrightarrow U$.

$\tau(U)$ = set of open covers of U

= set of families $\{U_i \xrightarrow{\varphi_i} U\} \text{ st } \bigcup_i \varphi_i(U_i) = U$.

If X scheme/S we define:

Small Zariski site: Objects $U \hookrightarrow X$

Morphisms $U' \rightarrow U$
 $\downarrow \quad \downarrow$
 $X \quad X$

Covering of U : $\{U_i \xrightarrow{\varphi_i} U\} \text{ st } \bigcup_i \varphi_i(U_i) = U$.

Small étale site: Objects: Étale morphisms $U \rightarrow X$, $U \in \text{Sch}/S$

Morphisms: $U \rightarrow U'$ étale
 $\downarrow \quad \downarrow$
 $X \quad X$

Covers of U : $\{U_i \xrightarrow{\varphi_i} U\}$ φ_i : étale st $\bigcup_i \varphi_i(U_i) = U$.

Same construction with étale replaced by:

- smooth
- fppf
- fpqc

Remark:

Open embedding \Rightarrow Étale \Rightarrow Smooth \Rightarrow fppf \Rightarrow fpqc

Big sites • Étale: Objects: Sch/S

Morphisms: morphisms of schemes

Covers of X : same.

Def: If \mathcal{C} is a site.

A presheaf on \mathcal{C} is $F: \mathcal{C}^{\text{op}} \rightarrow \text{Set}$

A sheaf on \mathcal{C} is a presheaf $F: \mathcal{C}^{\text{op}} \rightarrow \text{Set}$ such that

$\forall U \in \mathcal{C}$, $f, g \in F(U)$ if $\forall \{U_i \xrightarrow{\varphi_i} U\} \in \tau(U)$

$f|_{U_i} = g|_{U_i}$ (Here: $F(\varphi_i)(f) = F(\varphi_i)(g)$) $\Rightarrow f = g$.

2. $\forall u \in \mathcal{C}, \forall \{u_i \xrightarrow{\varphi_i} u\} \in \tau(u)$

$\forall f_i \in F(u_i) \text{ st } F(\varphi_{i,j,i})(f_i) = F(\varphi_{i,j,j})(f_j) \text{ in } F(u_i \times_u u_j)$

$[\varphi_{i,j,i} : u_i \times_u u_j \rightarrow u_i] \quad "u_i \cap u_j \subseteq u_i" \quad "u_i \cap u_j \subseteq u_j"$

$\Rightarrow \exists f \in F(u) \text{ st } \forall i \ f|_{u_i} = f_i.$

Remark: Sheaf for the fpqc top \Rightarrow sheaf for fppf \Rightarrow smooth \Rightarrow étale \Rightarrow Zar.

If $X \in \text{Sch}/S$ we can define $\text{Hom}(-, X) : (\text{Sch}/S)^{\text{op}} \rightarrow \text{Set}$

Thm: This is a sheaf for the Zariski topology. (tautological)

Thm (Grothendieck): This is a sheaf for the fpqc topology ("descent theory").

Remark: Schemes \subseteq Alg. spaces \subseteq Sheaves \subseteq Functors $\text{Sch}/S^{\text{op}} \rightarrow \text{Set}.$

II. 2-categories

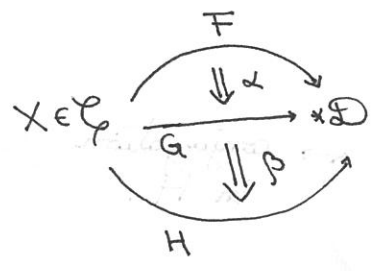
Cat = category of all categories

Objects: categories

Morphisms: functors.

+ between two functors $\mathcal{C} \xrightarrow{F} \mathcal{D} \xrightarrow{G}$ natural transformations

+ vertical composition of natural transformations



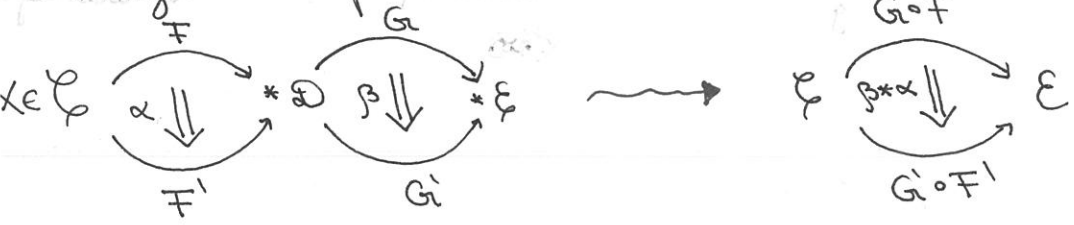
$\alpha_x : F(x) \rightarrow G(x)$

$\beta_x : G(x) \rightarrow H(x)$

$\beta \circ \alpha : F \Rightarrow H$

$(\beta \circ \alpha)_x : \beta_x \circ \alpha_x$

+ horizontal composition:



$\beta * \alpha : G \circ F \Rightarrow G' \circ F'$

$x \in \mathcal{C} : \alpha_x : F(x) \rightarrow F'(x)$

$G \alpha_x : GF(x) \rightarrow GF'(x)$

$\beta_{F'(x)} : G(F'(x)) \rightarrow G'(F'(x))$

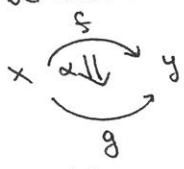
$\beta_{F'(x)} \circ G \alpha_x : GF(x) \rightarrow G'F'(x)$

Def: A 2-category \mathcal{C} is:

- \mathcal{C}_0 : Set of objects = Set of 0-morphisms
- \mathcal{C}_1 : Set of 1-morphisms
- \mathcal{C}_2 : Set of 2-morphisms

st $\mathcal{C}_0, \mathcal{C}_1$ categories.

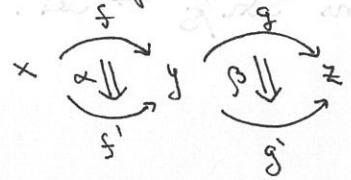
+ between two parallel 1-morphisms $x \begin{matrix} \xrightarrow{f} \\ \xrightarrow{g} \end{matrix} y$ there are 2-morphisms



+ identity 2-morphism

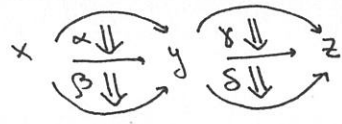
+ vertical composition of 2-morphisms, associativity, identity.

+ horizontal composition



$$\beta * \alpha : g \circ f \Rightarrow g' \circ f'$$

+ Axiom:

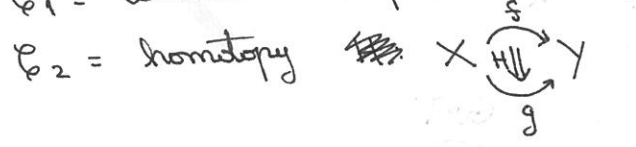


$$(\delta \circ \gamma) * (\beta \circ \alpha) = (\delta * \beta) \circ (\gamma * \alpha)$$

Examples:

- Cat
- Gpd: 1-morphisms are isomorphisms
- Grp (groups): Here: 2-morphism between $f, g: G \rightarrow H \iff$ conjugation in H .

- \mathcal{C} : $\mathcal{C}_0 =$ topological spaces
 $\mathcal{C}_1 =$ continuous maps $f: X \rightarrow Y$



modulo homotopies of $\left(\begin{matrix} \rightarrow \\ \rightarrow \end{matrix} \right)$ vertical composition is well def

$$H: X \times [0,1] \rightarrow Y$$

Other point of view:

A 2-category \iff Category \mathcal{C} st $\text{Hom}(x,y)$ is a category (with vertical comp. of 2-morph.)

$$\begin{matrix} \text{Hom}(y,z) \times \text{Hom}(x,y) & \rightarrow & \text{Hom}(x,z) \\ (g, f) & \mapsto & g \circ f \end{matrix}$$

is a lax functor (\approx distributivity \iff horizontal composition)

Functor between 2-categories : 2-functors.

All this is called strict 2-categories.

Weak versions :

Motivation : • A 1-morphism $f: x \rightarrow y$ is called invertible if $\exists g: y \rightarrow x$ st $gf = id_x, fg = id_y$.
• Same for invertible 2-morphism.

→ A 1-morphism $f: x \rightarrow y$ is called an equivalence if $\exists g: y \rightarrow x$ + invertible 2-morphisms $gf \cong id_x, fg \cong id_y$.

Example : Cat : Functor being an equivalence of categories
Top : Homotopy equivalence.

Recall : $F: \mathcal{C} \rightarrow \mathcal{D}$ is an equiv. of categories
 $\iff \forall y \in \mathcal{D} \exists x \in \mathcal{C}$ st $F(x) \cong y$
+ $F: Hom(a,b) \rightarrow Hom(F(a), F(b))$ is a bijection.

Def : A pseudo-functor (weak 2-functor) is $F: \mathcal{C} \rightarrow \mathcal{D}$ between 2-categories st:

$\forall x \in \mathcal{C}, \exists \alpha_x$ invertible 2-morphism with $F(id_x) \xrightarrow{\alpha_x} id_{F(x)}$
and $\forall f, g$ 1-morphisms $\exists \epsilon_{g,f}$ invertible 2-morphism st:
 $F(g \circ f) \xrightarrow{\epsilon_{g,f}} F(g) \circ F(f)$

+ F strictly associative on 2-morphisms
+ Axioms on α, ϵ .

Weak 2-category : 2-morphisms, associative composition
For 1-morphisms $h \circ (g \circ f) \cong (h \circ g) \circ f$.
↑
invertible 2-morphism

III. Stacks

Def: Let \mathcal{C} be any category. A pre-stack is a pseudo-functor $\mathcal{X} : \mathcal{C}^{\text{op}} \rightarrow \text{Gpd}$

Let \mathcal{C} be a site, a stack is a pseudo-functor $\mathcal{X} : \mathcal{C}^{\text{op}} \rightarrow \text{Gpd}$

[Notation: $x \in \mathcal{X}(U)$ and $f : U' \rightarrow U$, $x|_{U'} := \mathcal{X}(f)(x)$]

• Gluing of objects:

$$\forall U \in \mathcal{C}, \forall \{U_i \rightarrow U\} \in \tau(U), \forall x_i \in \mathcal{X}(U_i)$$

$$+ \text{morphisms } \varphi_{ij} : x_i|_{U_{ij}} \rightarrow x_j|_{U_{ij}} \text{ st } \varphi_{ij}|_{U_{ijk}} \circ \varphi_{jk}|_{U_{ijk}} = \varphi_{ik}|_{U_{ijk}}$$

$$\Rightarrow \exists x \in \mathcal{X}(U) \text{ and } \varphi_j : x|_{U_i} \xrightarrow{\sim} x_i \text{ in } \mathcal{X}(U_i)$$

$$\text{st } \varphi_{ji} \circ \varphi_i|_{U_{ij}} = \varphi_j|_{U_{ij}}$$

• Isom. form a sheaf:

$$\forall x, y \in \mathcal{X}(U), \varphi_i : x|_{U_i} \rightarrow y|_{U_i} \text{ st } \varphi_i|_{U_{ij}} = \varphi_j|_{U_{ij}}$$

$$\Rightarrow \exists \eta : x \rightarrow y \text{ st } \eta|_{U_i} = \varphi_i$$

$$\text{and } \forall x, y \in \mathcal{X}(U) \forall \varphi : x \rightarrow y, \psi : x \rightarrow y$$

$$\text{st } \varphi|_{U_i} = \psi|_{U_i} \Rightarrow \varphi = \psi$$

$U \mapsto \text{Isom}_U(x, y)$ is a sheaf.