

Lecture 3: "The 2-category of stacks" (7)

We shall discuss 3 ways of defining stacks:

- ① Via descent data
- ② Via categories fibered in groups
- ③ Via groupoid spaces

① Descent data: Let \mathcal{C} be a category.

Recall: A prestack is a pseudo functor $\mathcal{X} : \mathcal{C}^{op} \rightarrow \text{Grpds}$. In other words:

- 1) $\forall X \in \mathcal{C}$, $\mathcal{X}(X)$ is a groupoid (i.e. category in which all its morph. are isom.)
- 2) $\forall f \in \text{Hom}_{\mathcal{C}}(X, Y)$ there is a functor $f^* = \mathcal{X}(f) : \mathcal{X}(Y) \rightarrow \mathcal{X}(X)$ in Grpds
- 3) For each diagram $X \xrightarrow{f} Y \xrightarrow{g} Z$ in \mathcal{C} , $\exists \epsilon_{g,f} : (g \circ f)^* \cong f^* \circ g^*$ invertible nat. trans. in Grpds.

such that

$$\begin{array}{ccc} (h \circ g \circ f)^* & \xrightarrow{\cong} & (g \circ f)^* \circ h^* \\ \downarrow \epsilon & & \downarrow \epsilon \\ f^* \circ (h \circ g)^* & \xrightarrow{\cong} & f^* \circ g^* \circ h^* \end{array}$$

via the assoc. ϵ 's.

Def: Let \mathcal{C}_{τ} be a site and \mathcal{X} be a prestack. A descent datum for \mathcal{X} w.r.t. $\{U_i \xrightarrow{\varphi_{ij}} U_j\}_{i,j \in I}$ in \mathcal{C}_{τ} is a system $(x_i, \varphi_{ij})_{i,j \in I}$ st

- 1) $x_i \in \mathcal{X}(U_i)$
- 2) $\varphi_{ij} : x_i|_{U_{ij}} \rightarrow x_j|_{U_{ij}}$ morph. in $\mathcal{X}(U_{ij})$ st $\varphi_{ij}|_{U_{ijk}} \circ \varphi_{jk}|_{U_{ijk}} = \varphi_{ik}|_{U_{ijk}} \forall i,j,k \in I$

A descent datum is effective if $\exists x \in \mathcal{X}(U)$ and isomorphisms $\varphi_i : x|_{U_i} \xrightarrow{\cong} x_i$ in $\mathcal{X}(U_i) \forall i$ st $\varphi_{ji} \circ \varphi_i|_{U_{ij}} = \varphi_j|_{U_{ij}} \forall i,j \in I$.

Def (Stack): Let \mathcal{C}_{τ} be a site. A stack \mathcal{X} is a prestack st

- 1) Every descent datum for \mathcal{X} is effective.
- 2) $\forall U \in \mathcal{C}$ and $\forall x, y \in \mathcal{X}(U)$ the presheaf

$$\text{Isom}_U(x, y) : (\mathcal{C}/U)^{op} \rightarrow \text{Sets}$$

$$(U' \xrightarrow{\varphi} U) \mapsto \text{Hom}_{\mathcal{X}(U')} (f^* x, f^* y)$$

is a sheaf on the site $(\mathcal{C}/U)_{\tau}$.

→ "We can glue morphisms in a unique way"

Examples

1) (Sheaves): By def, if \mathcal{C}_{τ} is a site and $F : \mathcal{C}^{op} \rightarrow \text{Sets}$ is a sheaf on \mathcal{C}_{τ} , then \underline{F} is a stack (Here: $\underline{F}(X)$ groupoid w/ only morphisms $\text{id}_{\underline{F}(X)}$)

In gen, if $X \in \mathcal{C}$ then $\underline{X} = \text{Hom}_{\mathcal{C}}(-, X)$ is a prestack but not always a stack (not a sheaf).

Exam: If $\mathcal{C} = \text{Sch}/S$, $\tau = \text{fpqc}$ and $X \in \text{Sch}/S$ then \underline{X} is a sheaf, hence a stack (⇒ a stack for any coarser Grothendieck top).

→ \underline{X} is called the stack associated to the scheme X .

2) (Moduli stack of quasi-coherent sheaves): Let $X \in \text{Sch}/S$ and define $\mathcal{G}\text{coch}_X : (\text{Sch}/S)^{op} \rightarrow \text{Grpds}$

$$U \mapsto \left\{ \begin{array}{l} \text{quasi-coherent } \mathcal{O}_{X \times U}\text{-modules} \\ \text{flat over } U \\ + \text{ morphisms} = \text{isom. of } \mathcal{O}_{X \times U}\text{-modules.} \end{array} \right.$$

and $f : U' \rightarrow U \mapsto f^* : \mathcal{G}\text{coch}_X(U) \rightarrow \mathcal{G}\text{coch}_X(U')$ inverse image functor.

Fact (non-trivial): $\mathcal{G}\text{coch}_X$ is a stack for the fpqc topology. (see SGA1, Exposé VIII)

From this fact can be deduced that the moduli functors Coh_X (coherent sheaves), Bun_X^n (vect. bundles of rank n), $\text{Bun}_X^{n,d}$ (r.b. of rank n and deg d), $\text{Bun}_{S,m,d}$ (stable v.b...), $\text{Bun}_{S,m,d}^{\text{ss}}$ (semi-stable r.b...), and $\text{Bun}_{G,X}$ (principal G -bundles, G reductive alg group over a field k) are all stacks for the fppf topology on Sch/S .

3) (Quotient stack): Let $X \in \text{Sch}/S$ noetherian and G an affine smooth group S -scheme with right action $\rho: X \times G \rightarrow X$. Let us define the prestack $[X/G]: (\text{Sch}/S)^{\text{op}} \rightarrow \text{Groups}$

On objects: $U \mapsto [X/G](U) = \left\{ \begin{array}{l} \text{Principal } G\text{-bundles } \pi: E \rightarrow U \\ + G\text{-equivariant } \alpha: E \rightarrow X \end{array} \right\}$
 + morphisms = isom. of principal G -bundles commuting with G -equiv. α 's.

On morphisms: $(f: U' \rightarrow U) \mapsto f^*: [X/G](U) \rightarrow [X/G](U')$ induced by pullback of principal G -bundles.

Proposition: $[X/G]$ defines a stack on (Sch/S) w.r.t. the étale topology.

Idea of proof: Step 1 (descent datum): A principal bundle $E \rightarrow U$ is det. by étale descent (isom classes $\leftrightarrow H_{\text{ét}}^1(U, G)$).

Step 2 (Isom $_U(\alpha, \alpha')$ is a sheaf): Let $\alpha, \alpha' \in [X/G](U)$ corresp. to $\pi: E \rightarrow U, \alpha: E \rightarrow X$ and $\pi': E' \rightarrow U, \alpha': E' \rightarrow X$, resp.

Isom $_U(\alpha, \alpha'): (\text{Sch}/U)^{\text{op}} \rightarrow \text{Sets}$
 $(U' \xrightarrow{f} U) \mapsto \text{Hom}_{[X/G](U')} (f^* \alpha, f^* \alpha')$

$\Rightarrow \text{Isom}_U(\alpha, \alpha')$ is the étale sheaf (which is the quotient of $(X \times_X X \times_X E \times_U E')$ by the free product action of G).

Descent theory: this sheaf is in fact a scheme.

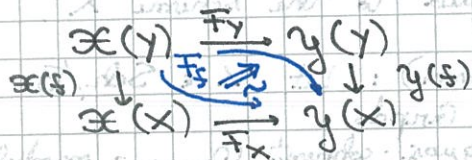
Remark: 1) If $X = S$ we get $\mathcal{B}G$ the classifying stack of the group S -scheme G .

2) This is true for flat affine group schemes (not nec. smooth).

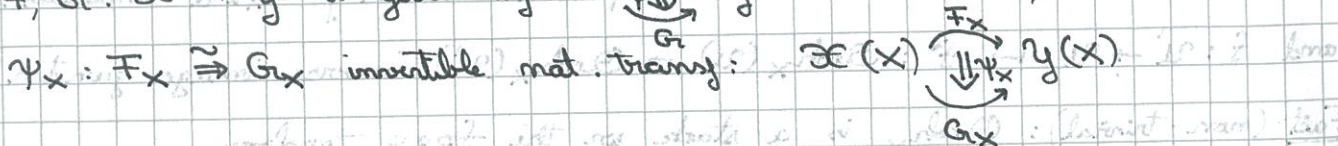
Stacks over \mathcal{C} form a 2-category.

Def (1-morphism): A 1-morphism of prestacks $F: \mathcal{X} \rightarrow \mathcal{Y}$ is a mat. transy. of functors of 2-categories. Namely

- $\forall X \in \mathcal{C}, F_X: \mathcal{X}(X) \rightarrow \mathcal{Y}(X)$
- $\forall f \in \text{Hom}_{\mathcal{C}}(X, Y)$ an invertible mat transy $F_f: \mathcal{Y}(f) \circ F_X \xrightarrow{\sim} F_Y \circ \mathcal{X}(f)$ compatible with $E_{g,f}: (g \circ f)^* \xrightarrow{\sim} f^* \circ g^*$.



Def (2-morphism): A 2-morphism between 1-morphisms of prestacks $F, G: \mathcal{X} \rightarrow \mathcal{Y}$ is given by $\mathcal{X} \begin{array}{c} \xrightarrow{F} \\ \Downarrow \Psi_X \\ \xrightarrow{G} \end{array} \mathcal{Y}$ st $\forall X \in \mathcal{C}$ there is



Prop: 1) Prestacks over \mathcal{C} form a 2-category $\text{PreStacks}(\mathcal{C})$.

2) Stacks over \mathcal{C}_τ form a 2-category $\text{Stacks}(\mathcal{C}_\tau)$.

Moreover, $\text{Stacks}(\mathcal{C}_\tau)$ is a full 2-subcat. of $\text{PreStacks}(\mathcal{C})$.

Remark: 1) $\mathcal{C} \subseteq \text{PreShv}(\mathcal{C}) \subseteq \text{PreStacks}(\mathcal{C})$ are full 2-subcats.

2) $\text{Shv}(\mathcal{C}_\tau) \subseteq \text{Stacks}(\mathcal{C}_\tau)$. In part, if every $X \in \mathcal{C}$ defines a sheaf $\underline{X} = \text{Hom}_{\mathcal{C}}(-, X)$ for $\tau \Rightarrow \mathcal{C} \subseteq \text{Stacks}(\mathcal{C}_\tau)$ as full 2-subcat.

Theorem (2-Yoneda): Let \mathcal{X} be a prestack over \mathcal{C} . Then $\forall X \in \mathcal{C}$

$$\Theta : \text{Hom}_{\text{PreStacks}}(\underline{X}, \mathcal{X}) \rightarrow \mathcal{X}(X) \quad \text{is an equivalence.}$$

$$(\mathbb{F} : \underline{X} \rightarrow \mathcal{X}) \mapsto \mathbb{F}(\text{id}_X)$$

Proof: Define the functor $\Xi : \mathcal{X}(X) \rightarrow \text{Hom}_{\text{PreStacks}}(\underline{X}, \mathcal{X})$

where \mathbb{F}_x sends $f \in X(U) = \text{Hom}_{\mathcal{C}}(U, X)$ to $f^*(x) \in \mathcal{X}(U)$.

One verify that $\Theta \circ \Xi = \text{id}_{\mathcal{X}(X)}$ and $\Xi \circ \Theta \cong \text{Id}_{\text{Hom}_{\text{PreStacks}}(\underline{X}, \mathcal{X})}$.

$\text{Stacks}(\mathcal{C}_\tau)$ has 2-fiber products:

Def: Let $\mathcal{X}, \mathcal{X}'$ and \mathcal{A} be stacks over \mathcal{C}_τ and $\mathbb{F} : \mathcal{X} \rightarrow \mathcal{A}, \mathbb{F}' : \mathcal{X}' \rightarrow \mathcal{A}$ morphisms of stacks. The stack $\mathcal{X} \times_{\mathcal{A}} \mathcal{X}'$ is defined by

$$U \in \mathcal{C} \mapsto (\mathcal{X} \times_{\mathcal{A}} \mathcal{X}') (U) \quad \text{groupoid with}$$

Objects: (u, u', ϕ) with $u \in \mathcal{X}(U), u' \in \mathcal{X}'(U), \phi \in \text{Hom}_{\mathcal{A}(U)}(\mathbb{F}(u), \mathbb{F}'(u'))$.

Morphisms: $\text{Hom}((u, u', \phi), (v, v', \psi)) = \{(u \xrightarrow{f} v, u' \xrightarrow{f'} v') : \psi \circ \mathbb{F}(f) = \phi \circ \mathbb{F}'(f')\}$.

Remark: 1) If $X, X' \in \mathcal{C}$ and $\alpha : X \rightarrow \mathcal{A}, \alpha' : X' \rightarrow \mathcal{A}$ morphisms of stacks. Then

$$\underline{X} \times_{\mathcal{A}} \underline{X}' \cong \text{Hom}_{\mathcal{A}(-)}(\text{pr}_1^* \alpha, \text{pr}_2^* \alpha') : (\mathcal{C} / X \times X')^{\text{op}} \rightarrow \text{Sets}$$

given by $(U \xrightarrow{h} X \times X') \mapsto \text{Hom}_{\mathcal{A}(U)}(h^* \text{pr}_1^* \alpha, h^* \text{pr}_2^* \alpha')$.

2) If $X \in \mathcal{C}$ and $\alpha : X \rightarrow \mathcal{X}$ morphism of stacks. Then,

$$\underline{X} \times_{\mathcal{X}} \mathcal{X} \cong \text{Isom}_X(\alpha, \alpha) : (\mathcal{C} / X)^{\text{op}} \rightarrow \text{Sets}$$

$$(U \xrightarrow{f} X) \mapsto \text{Hom}_{\mathcal{X}(U)}(f^* \alpha, f^* \alpha)$$

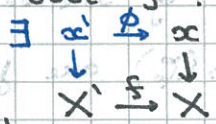
sheaf on \mathcal{C}/X .

② Categories fibered in groupoids: Let \mathcal{C} be a category.

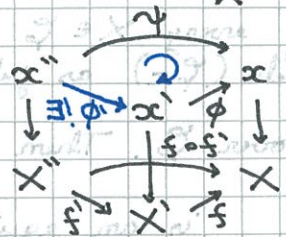
Def: A category fibered over \mathcal{C} is a category \mathcal{X} with a projection functor $\mathbb{P}_{\mathcal{X}} : \mathcal{X} \rightarrow \mathcal{C}$.

Notation: If $x \in \mathcal{X}$ with $\mathbb{P}_{\mathcal{X}}(x) = X$ we say that "x lies over X" similar for morphisms.

Def: A category fibered in groupoids is $\mathcal{P}_{\mathcal{X}}: \mathcal{X} \rightarrow \mathcal{C}$ st $\forall f: X' \rightarrow X$ in \mathcal{C} and $\alpha \in \mathcal{X}$ over X , $\exists \alpha'$ over X' and $\phi: \alpha' \rightarrow \alpha$ over f .



2) Given



st $\exists \phi': \alpha' \rightarrow \alpha$ st $\phi = \phi' \circ f'$

Rmk. 2) implies that α' in 1) is unique up to a canonical isom (we can choose $\alpha' = f^* \alpha$ therefore)
 ϕ isom $\iff f$ isom in 1)

For $X \in \mathcal{C}$ we define the fiber category

$$\mathcal{X}(X) = \{ \alpha \in \mathcal{X} \text{ st } \mathcal{P}_{\mathcal{X}}(\alpha) = X, \phi: \alpha \rightarrow \alpha \text{ st } \mathcal{P}_{\mathcal{X}}(\phi) = \text{id}_X \}$$

This is a groupoid as all ϕ over id_X are isomorphisms.

There is an equivalence

$$\{ \text{Prestacks over } \mathcal{C} \} \longleftrightarrow \{ \text{Categories fibered in groupoids over } \mathcal{C} \}$$

$$\tilde{\mathcal{X}}: \mathcal{C}^{\text{op}} \rightarrow \text{Grpds} \longleftrightarrow \mathcal{P}_{\mathcal{X}}: \mathcal{X} \rightarrow \mathcal{C}$$

$$X \mapsto \mathcal{X}(X)$$

$\tilde{\mathcal{X}}: \mathcal{C}^{\text{op}} \rightarrow \text{Grpds} \longmapsto \mathcal{X}$ with Objects: (α, X) , $X \in \mathcal{C}$ and $\alpha \in \mathcal{X}(X)$

Morphisms: $(\alpha', X') \rightarrow (\alpha, X)$ given by (f, α) where $f: X' \rightarrow X$ and $\alpha: f^* \alpha \rightarrow \alpha$ isom. with $f^* = \tilde{\mathcal{X}}(f)$.

Def (Stack): Let $\mathcal{C} \tau$ be a site. A stack \mathcal{X} is a category fibered in groupoids over \mathcal{C} st

- 1) $\forall U \in \mathcal{C}$ and $\forall \alpha, \beta \in \mathcal{X}$ lying over U , the presheaf $\text{Inom}_U(\alpha, \beta): (\mathcal{C}/U)^{\text{op}} \rightarrow \text{Sets}$ is a sheaf on the site $(\mathcal{C}/U) \tau$.
- 2) Every descent datum is effective.

With this definition we can rephrase:

- a) (1-morphism) $F: \mathcal{X} \rightarrow \mathcal{Y}$ functor between cat. fibered in groupoids over \mathcal{C} st $\mathcal{P}_{\mathcal{Y}} \circ F = \mathcal{P}_{\mathcal{X}}$. In part, $\mathcal{X} \cong \mathcal{Y}$ $\iff \exists$ 1-morphism $F: \mathcal{X} \rightarrow \mathcal{Y}$ st F equiv. of categories.
- b) (2-morphism) $\psi: F \Rightarrow G$ natural transy. over the identity functor $\text{id}_{\mathcal{C}}$.
- c) (Stack associated to an object): Let $X \in \mathcal{C}$, the stack \underline{X} is \mathcal{C}/X with $\mathcal{P}_X: \mathcal{C}/X \rightarrow \mathcal{C}$, $(U \xrightarrow{f} X) \mapsto U$.

d) (Fibered product): Let $\mathcal{X}, \mathcal{X}'$ and Δ stacks and $F: \mathcal{X} \rightarrow \Delta$, $F': \mathcal{X}' \rightarrow \Delta$ morphisms. We define the cat. fibered in groupoids $\mathcal{X} \times_{\Delta} \mathcal{X}'$ over \mathcal{C} and projection functors $p_{\mathcal{X}}$ and $p_{\mathcal{X}'}$ with

Objects: (x, x', α) with $x \in \mathcal{X}, x' \in \mathcal{X}'$ both over the same X and $\alpha: F(x) \rightarrow F'(x')$ st $\varphi_{\Delta}(\alpha) = \text{id}_X$.

Morphisms: $(\phi, \phi'): (x, x', \alpha) \rightarrow (y, y', \beta)$ with $\phi: x \rightarrow y, \phi': x' \rightarrow y'$ both over the same $f: X \rightarrow Y$ in \mathcal{C} st $\beta \circ F(\phi) = F'(\phi') \circ \alpha$.

③ Groupoid spaces: Let \mathcal{C} be a category with fiber products.

Def: A groupoid $[X_1 \rightrightarrows X_0]$ in \mathcal{C} consists of sets X_0 (set of objects), X_1 (set of morphisms) and five maps of sets s, t, e, m, i given as:

$$X_1 \begin{array}{c} \xrightarrow{s} \\ \xleftarrow{t} \end{array} X_0 \xrightarrow{e} X_1$$

$$X_1 \times_{s, X_0, t} X_1 \xrightarrow{m} X_1, \quad X_1 \xrightarrow{i} X_1 \text{ satisfying:}$$

- 1) (Compatibility) $se = te = \text{id}_{X_1}, si = t, ti = s, sm = spr_2, tm = tpr_1$
- 2) (Associativity) $m(m \times \text{id}_{X_1}) = m(\text{id}_{X_1} \times m)$
- 3) (Identity) $m(\text{id}_{X_1} \times e) = m(e \times \text{id}_{X_1}) = \text{id}_{X_1}$
- 4) (Inverse) $m(i \times \text{id}_{X_1}) = es, m(\text{id}_{X_1} \times i) = et$.

Def: Let \mathcal{C}_{τ} be a site. A groupoid space is a groupoid $[X_1 \rightrightarrows X_0]$ in the category $\text{Shv}(\mathcal{C}_{\tau})$.

Fact: There is an equiv. between the 2-categories

$$\{ \text{Groupoid spaces over } \mathcal{C}_{\tau} \} \longleftrightarrow \{ \text{Categories fibered in groupoids over } \mathcal{C}_{\tau} \}$$

$$[X_1 \rightrightarrows X_0] \longmapsto [X_1 \rightrightarrows X_0] \text{ category with:}$$

Objects over $U \in \mathcal{C}$: $X_0(U)$

Morphisms over id_U : $X_1(U)$

Remark: In genl, $[X_1 \rightrightarrows X_0]$ is only a prestack, but it is possible to "stackify" this to obtain a stack $[X_0/X_1]$ associated to $[X_1 \rightrightarrows X_0]$.

We can use this construction to define "stacky quotients" $[X/R]$, where $X: (\text{Sch}/S)^{\text{op}} \rightarrow \text{Sets}$ is a sheaf and R an equiv. relation.

See Laumon & Moret-Bailly "Champs algébriques" Chapter 2 for more details.