Lecture 3: ~~[illegible]~~ "The 2-category of stacks"

We shall discuss 3 ways of defining stacks:
1. Via descent data
2. Via categories fibered in grpds
3. Via groupoid spaces

1. Descent data: Let $\mathcal{C}$ be a category.

Recall: A prestack is a pseudo functor $\mathcal{X} : \mathcal{C}^{op} \to \mathrm{Grpds}$. In other words:
1) $\forall X \in \mathcal{C}$, $\mathcal{X}(X)$ is a groupoid (ie, category in which all its morph. are isom)
2) $\forall f \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$ there is a functor $f^* = \mathcal{X}(f) : \mathcal{X}(Y) \to \mathcal{X}(X)$ in Grpds.
3) For each diagram $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{C}$, $\exists\, \varepsilon_{g,f} : (g \circ f)^* \Rightarrow f^* \circ g^*$ invertible nat. transf. in Grpds.

such that

$$\begin{tikzcd} (h \circ g \circ f)^* \arrow[r, Rightarrow, "\sim"] \arrow[d, Rightarrow, "S"] & (g \circ f)^* \circ h^* \arrow[d, Rightarrow, "S"] \\ f^* \circ (h \circ g)^* \arrow[r, Rightarrow, "\sim"] & f^* \circ g^* \circ h^* \end{tikzcd}$$

via the corresp. ε's.

Def: Let $\mathcal{C}_\tau$ be a site and $\mathcal{X}$ be a prestack. A descent datum for $\mathcal{X}$ wrt $\{\mathcal{U}_i \xrightarrow{f_i} \mathcal{U}\}_{i \in I}$ in $\mathcal{C}_\tau$ is a system $(x_i, \varphi_{ij})_{i,j \in I}$ st
1) $x_i \in \mathcal{X}(\mathcal{U}_i)$
2) $\varphi_{ij} : x_i|_{\mathcal{U}_{ij}} \to x_j|_{\mathcal{U}_{ij}}$ morph. in $\mathcal{X}(\mathcal{U}_{ij})$ st $\varphi_{ij}|_{\mathcal{U}_{ijk}} \circ \varphi_{jk}|_{\mathcal{U}_{ijk}} = \varphi_{ik}|_{\mathcal{U}_{ijk}}$ $\forall i,j,k \in I$

A descent datum is effective if $\exists\, x \in \mathcal{X}(\mathcal{U})$ and isomorphisms $\varphi_i : x|_{\mathcal{U}_i} \xrightarrow{\sim} x_i$ in $\mathcal{X}(\mathcal{U}_i)$ $\forall i$ st $\varphi_{ji} \circ \varphi_i|_{\mathcal{U}_{ij}} = \varphi_j|_{\mathcal{U}_{ij}}$ $\forall i,j \in I$.

Def (Stack): Let $\mathcal{C}_\tau$ be a site. A stack $\mathcal{X}$ is a prestack st
1) Every descent datum for $\mathcal{X}$ is effective.
2) $\forall \mathcal{U} \in \mathcal{C}$ and $\forall x,y \in \mathcal{X}(\mathcal{U})$ the presheaf

$$\begin{aligned} \mathrm{Isom}_{\mathcal{U}}(x,y) : (\mathcal{C}/\mathcal{U})^{op} &\to \mathrm{Sets} \\ (\mathcal{U}' \xrightarrow{f} \mathcal{U}) &\mapsto \mathrm{Hom}_{\mathcal{X}(\mathcal{U}')}(f^* x, f^* y) \end{aligned}$$

→ "We can glue morphisms in a unique way"

is a sheaf on the site $(\mathcal{C}/\mathcal{U})_\tau$.

Examples

1) (Sheaves): By def, if $\mathcal{C}_\tau$ is a site and $F : \mathcal{C}^{op} \to \mathrm{Sets}$ is a sheaf on $\mathcal{C}_\tau$, then F is a stack (Here: $F(X)$ groupoid w/ only morphisms $id_{F(X)}$)

In gral, if $X \in \mathcal{C}$ then $\underline{X} = \mathrm{Hom}_{\mathcal{C}}(-, X)$ is a prestack but not always a stack (not a sheaf).

Seen: If $\mathcal{C} = Sch/S$, $\tau = fpqc$ and $X \in Sch/S$ then $\underline{X}$ is a sheaf, hence a stack ($\Rightarrow$ a stack for any coarser Grothendieck top).

→ $\underline{X}$ is called the stack associated to the scheme X.

2) (Moduli stack of quasi-coherent sheaves): Let $X \in Sch/S$ and define

$$\begin{aligned} \mathcal{Q}coh_X : (Sch/S)^{op} &\to \mathrm{Grpds} \\ \mathcal{U} &\mapsto \left\{ \begin{array}{l} \text{quasi-coherent } \mathcal{O}_{X \times \mathcal{U}}\text{-modules} \\ \text{flat over } \mathcal{U} \end{array} \right\} \end{aligned}$$

+ morphisms = isom. of $\mathcal{O}_{X \times \mathcal{U}}$-modules.

and $f : \mathcal{U}' \to \mathcal{U} \mapsto f^* : \mathcal{Q}coh_X(\mathcal{U}) \to \mathcal{Q}coh_X(\mathcal{U}')$ inverse image functor.

Fact (non-trivial): $\mathcal{Q}coh_X$ is a stack for the fpqc topology.
(see SGA 1, Exposé VIII)

From this fact can be deduced that the moduli functors Coh_X (coherent sheaves), Bun_X^n (vect. bundles of rank n), $\mathrm{Bun}_C^{n,d}$ (v.b. of rk n and deg d), $\mathrm{Bun}_C^{s,n,d}$ (stable v.b...), $\mathrm{Bun}_C^{ss,n,d}$ (semi-stable v.b...) and $\mathrm{Bun}_{G,X}$ (principal G-bundles, G reductive alg group over a field k) are all stacks for the fpqc topology on Sch/S.

3) (Quotient stack): Let $X \in \mathrm{Sch}/S$ noetherian and G an affine smooth group S-scheme with right action $\rho : X \times G \to G$. Let us define the prestack $[X/G] : (\mathrm{Sch}/S)^{op} \to \mathrm{Grpds}$

<u>On objects</u>: $\mathcal{U} \mapsto [X/G](\mathcal{U}) = \left\{ \begin{array}{l} \text{Principal } G\text{-bundles } \pi : \mathcal{E} \to \mathcal{U} \\ + G\text{-equivariant } \alpha : \mathcal{E} \to X \end{array} \right\}$
+ morphisms = isom. of principal G-bundles commuting with G-equiv α's.

<u>On morphisms</u>: $(f : \mathcal{U}' \to \mathcal{U}) \mapsto f^* : [X/G](\mathcal{U}) \to [X/G](\mathcal{U}')$ induced by pullback of principal G-bundles.

<u>Proposition</u>: $[X/G]$ defines a stack on (Sch/S) wrt the étale topology.

<u>Idea of proof</u>: Step 1 (descent datum): A principal bundle $\mathcal{E} \to \mathcal{U}$ is det. by étale descent (isom classes $\leftrightarrow H^1_{et}(\mathcal{U}, G)$).

<u>Step 2</u> ($\mathrm{Isom}_{\mathcal{U}}(x, x')$ is a sheaf): Let $x, x' \in [X/G](\mathcal{U})$ corresp. to $\pi : \mathcal{E} \to \mathcal{U}$, $\alpha : \mathcal{E} \to X$ and $\pi' : \mathcal{E}' \to \mathcal{U}$, $\alpha' : \mathcal{E}' \to X$, resp.

$\mathrm{Isom}_{\mathcal{U}}(x, x') : (\mathrm{Sch}/\mathcal{U})^{op} \to \mathrm{Sets}$
$(\mathcal{U}' \xrightarrow{f} \mathcal{U}) \mapsto \mathrm{Hom}_{[X/G](\mathcal{U}')}(f^* x, f^* x')$

$\Rightarrow \mathrm{Isom}_{\mathcal{U}}(x, x')$ is the étale sheaf (which is the quotient of $(X \times_{X \times X} \mathcal{E} \times_{\mathcal{U}} \mathcal{E}')$ by the free product action of G).
Descent theory: this sheaf is in fact a scheme ■

<u>Rmk</u>: 1) If $X = S$ we get BG the classifying stack of the group S-scheme G.
2) This is true for flat affine group schemes (not nec. smooth).

Stacks over $\mathcal{C}_\tau$ form a 2-category.

<u>Def</u> (1-morphism): A 1-morphism of prestacks $F : \mathcal{X} \to \mathcal{Y}$ is a nat. transf. of functors of 2-categories. Namely,
1) $\forall X \in \mathcal{C}$, $F_X : \mathcal{X}(X) \to \mathcal{Y}(X)$
2) $\forall f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ an invertible nat transf $F_f : \mathcal{Y}(f) \circ F_X \xRightarrow{\sim} F_Y \circ \mathcal{X}(f)$ compatible with $\varepsilon_{g,f} : (g \circ f)^* \Rightarrow f^* \circ g^*$:

$$\begin{tikzcd} \mathcal{X}(Y) \arrow[r, "F_Y"] \arrow[d, "\mathcal{X}(f)"'] \arrow[dr, phantom, "F_f \Uparrow\wr"] & \mathcal{Y}(Y) \arrow[d, "\mathcal{Y}(f)"] \\ \mathcal{X}(X) \arrow[r, "F_X"'] & \mathcal{Y}(X) \end{tikzcd}$$

<u>Def</u> (2-morphism): A 2-morphism between 1-morphisms of prestacks $F, G : \mathcal{X} \to \mathcal{Y}$ is given by $\mathcal{X} \underset{G}{\overset{F}{\Downarrow\psi}} \mathcal{Y}$ st $\forall X \in \mathcal{C}$ there is $\psi_X : F_X \xRightarrow{\sim} G_X$ invertible nat. transf:

$$\begin{tikzcd} \mathcal{X}(X) \arrow[r, bend left, "F_X"] \arrow[r, bend right, "G_X"'] \arrow[r, phantom, "\Downarrow\psi_X"] & \mathcal{Y}(X) \end{tikzcd}$$

Prop: 1) Prestacks over $\mathcal{C}$ form a 2-category PreStacks($\mathcal{C}$).

2) Stacks over $\mathcal{C}_\tau$ form a 2-category Stacks($\mathcal{C}_\tau$).

Moreover, Stacks($\mathcal{C}_\tau$) is a full 2-subcat. of PreStacks($\mathcal{C}$). ■

Remark: 1) $\mathcal{C} \subseteq \mathrm{PreShv}(\mathcal{C}) \subseteq \mathrm{PreStacks}(\mathcal{C})$ are full 2-subcats.

2) $\mathrm{Shv}(\mathcal{C}_\tau) \subseteq \mathrm{Stacks}(\mathcal{C}_\tau)$. In part, if every $X \in \mathcal{C}$ defines a sheaf $\underline{X} = \mathrm{Hom}_{\mathcal{C}}(-, X)$ for $\tau \Rightarrow \mathcal{C} \subseteq \mathrm{Stacks}(\mathcal{C}_\tau)$ as full 2-subcat.

Theorem (2-Yoneda): Let $\mathfrak{X}$ be a prestack over $\mathcal{C}$. Then $\forall X \in \mathcal{C}$

$$\Theta : \mathrm{Hom}_{\mathrm{PreStacks}}(\underline{X}, \mathfrak{X}) \to \mathfrak{X}(X) \quad \text{is an equivalence.}$$
$$(F : \underline{X} \to \mathfrak{X}) \mapsto F(\mathrm{id}_X)$$

Proof: Define the functor $\Xi : \mathfrak{X}(X) \to \mathrm{Hom}_{\mathrm{PreStacks}}(\underline{X}, \mathfrak{X})$
$$x \mapsto F_x : \underline{X} \to \mathfrak{X}$$
where F_x sends $f \in \underline{X}(U) = \mathrm{Hom}_{\mathcal{C}}(U, X)$ to $f^*(x) \in \mathfrak{X}(U)$.

One verify that $\Theta \circ \Xi = \mathrm{id}_{\mathfrak{X}(X)}$ and $\Xi \circ \Theta \overset{\sim}{\Rightarrow} \mathrm{Id}_{\mathrm{Hom}_{\mathrm{PreStacks}(\mathcal{C})}(\underline{X}, \mathfrak{X})}$ ■

Stacks($\mathcal{C}_\tau$) has 2-fiber products:

Def: Let $\mathfrak{X}$, $\mathfrak{X}'$ and $\mathcal{S}$ be stacks over $\mathcal{C}_\tau$ and $F : \mathfrak{X} \to \mathcal{S}$, $F' : \mathfrak{X}' \to \mathcal{S}$ morphisms of stacks. The stack $\mathfrak{X} \times_{\mathcal{S}} \mathfrak{X}'$ is defined by

$$U \in \mathcal{C} \mapsto (\mathfrak{X} \times_{\mathcal{S}} \mathfrak{X}')(U) \quad \text{groupoid with}$$

Objects: (u, u', ϕ) with $u \in \mathfrak{X}(U)$, $u' \in \mathfrak{X}'(U)$, $\phi \in \mathrm{Hom}_{\mathcal{S}(U)}(F(u), F'(u'))$.

Morphisms: $\mathrm{Hom}((u, u', \phi), (v, v', \psi)) = \{(u \xrightarrow{f} v, u' \xrightarrow{f'} v') : \psi \circ F(f) = \phi \circ F'(f')\}$.

Remark: 1) If $X, X' \in \mathcal{C}$ and $x : \underline{X} \to \mathcal{S}$, $x' : \underline{X}' \to \mathcal{S}$ morphisms of stacks. Then,
$$\underline{X} \times_{\mathcal{S}} \underline{X}' \simeq \mathrm{Hom}_{\mathcal{S}(-)}(pr_1^* x, pr_2^* x') : (\mathcal{C}/X \times X')^{op} \to \mathrm{Sets}$$
given by $(U \xrightarrow{h} X \times X') \mapsto \mathrm{Hom}_{\mathcal{S}(U)}(h^* pr_1^* x, h^* pr_2^* x')$.

2) If $X \in \mathcal{C}$ and $x : \underline{X} \to \mathfrak{X}$ morphism of stacks. Then,
$$\underline{X} \times_{\mathfrak{X} \times \mathfrak{X}} \mathfrak{X} \simeq \mathrm{Isom}_X(x, x) : (\mathcal{C}/X)^{op} \to \mathrm{Sets}$$
$$(U \xrightarrow{f} X) \mapsto \mathrm{Hom}_{\mathfrak{X}(U)}(f^* x, f^* x).$$
sheaf on $\mathcal{C}/X$.

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(2) Categories fibered in groupoids: Let $\mathcal{C}$ be a category.

Def: A category fibered over $\mathcal{C}$ is a category $\mathfrak{X}$ with a projection functor $p_{\mathfrak{X}} : \mathfrak{X} \to \mathcal{C}$.

Notation: If $x \in \mathfrak{X}$ with $p_{\mathfrak{X}}(x) = X$ we say that "x lies over X". Similar for morphisms.

Def: A <u>category fibered in groupoids</u> is $p_{\mathfrak{X}} : \mathfrak{X} \to \mathcal{C}$ st
1) $\forall f : X' \to X$ in $\mathcal{C}$ and $x \in \mathfrak{X}$ over X, $\exists\, x'$ over X' and $\phi : x' \to x$ over f:

$$\begin{tikzcd} \exists\, x' \arrow[r, "\phi"] \arrow[d] & x \arrow[d] \\ X' \arrow[r, "f"] & X \end{tikzcd}$$

2) Given

$$\begin{tikzcd} x'' \arrow[rr, bend left, "\psi"] \arrow[dr, "\exists!\,\phi'"] \arrow[dd] & & x \arrow[dd] \\ & x' \arrow[ur, "\phi"] & \\ X'' \arrow[rr, "f \circ f'"] \arrow[dr, "f'"'] & & X \\ & X' \arrow[ur, "f"'] & \end{tikzcd}$$

$\exists!\ \phi' : x'' \to x'$ st
a) $\psi = \phi \circ \phi'$
b) $p_{\mathfrak{X}}(\phi') = f'$

Rmk: 2) implies that
- x' in 1) is unique up to a canonical isom (we can choose $x' = f^* x$ therefore)
- ϕ isom $\iff$ f isom in 1).

For $X \in \mathcal{C}$ we define the fiber category

$$\mathfrak{X}(X) = \{ x \in \mathfrak{X} \text{ st } p_{\mathfrak{X}}(x) = X,\ \phi : x \to x \text{ st } p_{\mathfrak{X}}(\phi) = id_X \}$$

This is a <u>groupoid</u> as all ϕ over id_X are isomorphisms.

There is an equivalence

$$\{\text{Prestacks over } \mathcal{C}\} \longleftrightarrow \{\text{Categories fibered in groupoids over } \mathcal{C}\}$$

$\tilde{\mathfrak{X}} : \mathcal{C}^{op} \to \text{Grpds}$, $X \mapsto \mathfrak{X}(X)$ $\longleftrightarrow$ $p_{\mathfrak{X}} : \mathfrak{X} \to \mathcal{C}$

$\tilde{\mathfrak{X}} : \mathcal{C}^{op} \to \text{Grpds}$ $\longmapsto$ $\mathfrak{X}$ with <u>Objects</u>: (x, X), $X \in \mathcal{C}$ and $x \in \tilde{\mathfrak{X}}(X)$

<u>Morphisms</u>: $(x', X') \to (x, X)$ given by (f, α) where $f : X' \to X$, $\alpha : f^* x \xrightarrow{\sim} x$ isom. with $f^* = \tilde{\mathfrak{X}}(f)$.

Def (Stack): Let $\mathcal{C}_\tau$ be a site. A <u>stack</u> $\mathfrak{X}$ is a category fibered in groupoids over $\mathcal{C}$ st
1) $\forall\, \mathcal{U} \in \mathcal{C}$ and $\forall x, y \in \mathfrak{X}$ lying over $\mathcal{U}$, the presheaf
$$\text{Isom}_{\mathcal{U}}(x, y) : (\mathcal{C}/\mathcal{U})^{op} \to \text{Sets}$$
is a sheaf on the site $(\mathcal{C}/\mathcal{U})_\tau$.
2) Every descent datum is effective.

With this definition we can rephrase:

a) (1-morphism) $F : \mathfrak{X} \to \mathcal{Y}$ functor between cat. fibered in groupoids over $\mathcal{C}$ st $p_{\mathcal{Y}} \circ F = p_{\mathfrak{X}}$. In part, $\mathfrak{X} \simeq \mathcal{Y}$ if $\exists$ 1-morphism $F : \mathfrak{X} \to \mathcal{Y}$ st F equiv. of categories.

b) (2-morphism) $\psi : F \Rightarrow G$ natural transf. over the identity functor $id_{\mathcal{C}}$.

c) (Stack associated to an object): Let $X \in \mathcal{C}$, the stack $\underline{X}$ is $\mathcal{C}/X$ with $p_X : \mathcal{C}/X \to \mathcal{C}$, $(\mathcal{U} \xrightarrow{f} X) \mapsto \mathcal{U}$.

d) (Fibered product): Let $\mathcal{X}$, $\mathcal{X}'$ and $\mathcal{S}$ stacks and $F: \mathcal{X} \to \mathcal{S}$, $F': \mathcal{X}' \to \mathcal{S}$ morphisms. We define the cat. fibered in grupoids $\mathcal{X} \times_{\mathcal{S}} \mathcal{X}'$ over $\mathcal{C}$ and projection functors $p_{\mathcal{X}}$ and $p_{\mathcal{X}'}$ with

Objects: (x, x', α) with $x \in \mathcal{X}$, $x' \in \mathcal{X}'$ both over the same X and $\alpha: F(x) \to F'(x')$ st $\varphi_{\mathcal{S}}(\alpha) = id_X$.

Morphisms: $(\phi, \phi'): (x, x', \alpha) \to (y, y', \beta)$ with $\phi: x \to y$, $\phi': x' \to y'$ both over the same $f: X \to Y$ in $\mathcal{C}$ st $\beta \circ F(\phi) = F'(\phi') \circ \alpha$.

③ Grupoid spaces: Let $\mathcal{C}$ be a category with fiber products.

Def: A grupoid $[X_1 \rightrightarrows X_0]$ in $\mathcal{C}$ consists of sets X_0 (set of objects), X_1 (set of morphisms) and five maps of sets s, t, e, m, i given as:

$$X_1 \underset{t}{\overset{s}{\rightrightarrows}} X_0 \xrightarrow{e} X_1$$

$$X_1 \times_{s, X_0, t} X_1 \xrightarrow{m} X_1 \quad , \quad X_1 \xrightarrow{i} X_1 \quad \text{satisfying:}$$

1) (Compatibility) $se = te = id_{X_1}$, $si = t$, $ti = s$, $sm = s\,pr_2$, $tm = t\,pr_1$
2) (Associativity) $m(m \times id_{X_1}) = m(id_{X_1}(m))$
3) (Identity) $m(id_{X_1} \times e) = m(e \times id_{X_1}) = id_{X_1}$
4) (Inverse) $m(i \times id_{X_1}) = es$, $m(id_{X_1} \times i) = et$.

Def: Let $\mathcal{C}_\tau$ be a site. A grupoid space is a grupoid $[X_1 \rightrightarrows X_0]$ in the category $Shv(\mathcal{C}_\tau)$.

Fact: There is an equiv. between the 2-categories

$$\{\text{Grupoid spaces over } \mathcal{C}_\tau\} \longleftrightarrow \{\text{Categories fibered in grupoids over } \mathcal{C}_\tau\}$$

$$[X_1 \rightrightarrows X_0] \longmapsto \widetilde{[X_1 \rightrightarrows X_0]} \quad \text{category with:}$$

Objects over $U \in \mathcal{C}$: $X_0(U)$

Morphisms over id_U: $X_1(U)$

Rmk: In gral, $\widetilde{[X_1 \rightrightarrows X_0]}$ is only a prestack, but it is possible to "stackify" this to obtain a stack $[X_0/X_1]$ associated to $[X_1 \rightrightarrows X_0]$.

We can use this construction to define "stacky quotients" $[X/R]$, where $X: (Sch/S)^{op} \to Sets$ is a sheaf and R an equiv. relation.

See Laumon & Moret-Bailly "Champs algébriques" Chapter 2 for more details.