Lecture 4 : "Fibered categories and algebraic stacks".

Fix $\mathcal{C}$ any category.

Def : A category over $\mathcal{C}$ is a category $\mathcal{X}$ with a functor $p : \mathcal{X} \to \mathcal{C}$. Then $\forall\, \mathcal{U} \in \mathcal{C}$ we consider $\mathcal{X}(\mathcal{U})$ to be the category with objects over $\mathcal{U}$ and morphisms over $id_{\mathcal{U}}$: $x \in \mathcal{X}$ st $p(x) = \mathcal{U}$,

$f : x \to y$ st $p(f) = id_{\mathcal{U}}$.

Def : A category fibered in groupoids is a category $\mathcal{X}$ over $\mathcal{C}$ st

(1) $\forall\, f : V \to \mathcal{U}$ in $\mathcal{C}$ and $\forall\, x \in \mathcal{X}(\mathcal{U})$, $\exists\, y \to x$ st $p(y \to x) = f$.

$$\begin{tikzcd} \exists\, y \arrow[r, dashed] \arrow[d, mapsto, "p"'] & x \arrow[d, mapsto] \\ V \arrow[r] & \mathcal{U} \end{tikzcd}$$

(2) Given

$$\begin{tikzcd} z \arrow[dr, dashed, "\exists !"] \arrow[drr, bend left] \arrow[dd, mapsto] & & \\ & y \arrow[r] \arrow[dd, mapsto] & x \arrow[dd, mapsto] \\ W \arrow[dr] \arrow[drr, bend left] & & \\ & V \arrow[r] & \mathcal{U} \end{tikzcd}$$

In this case, $\forall\, \mathcal{U}$ $\mathcal{X}(\mathcal{U})$ is a groupoid.

Examples :

1) Consider the prestack $[X/G]$. We associate to it the category $\mathcal{X}$ with :

Objects : All G-bundles $\begin{tikzcd} P \arrow[r, "G" loop above] \arrow[d] & X \\ \mathcal{U} & \end{tikzcd}$ [Here: $p : \mathcal{X} \to \mathcal{C}$, $\begin{pmatrix} P \\ \downarrow \\ \mathcal{U} \end{pmatrix} \mapsto \mathcal{U}$]

Morphisms : morphisms of principal bundles.

Then the axiom :

(1) says that $\exists$ pull-back of principal bundles.

(2) Take $W = V \rightsquigarrow$ universal property of pull-back.

2) Prestack of vector bundles of rank n over X.

We associate it the category $\mathcal{X}$ with

Objects : All vector bundles over $X \times \mathcal{U}$ [Here: $p : \mathcal{X} \to \mathcal{C}$, $(\mathcal{E} \to X \times \mathcal{U}) \mapsto \mathcal{U}$]

Morphisms : Over $V \to \mathcal{U}$: morphism of vector bundles

$$\begin{tikzcd} \mathcal{F} \arrow[r] \arrow[d] & \mathcal{G} \arrow[d] \\ X \times V \arrow[r] & X \times \mathcal{U} \end{tikzcd}$$

3) Prestack associated to an object $X \in \mathcal{C}$ ⇝ we assoc. the category $\mathcal{X} : (\mathcal{C}/X)^{op}$ with

Objects: $\mathcal{U} \to X$ + Morphisms (as. usual)

$$\left[\text{Here: } \begin{array}{l} p : \mathcal{X} \to \mathcal{C} \\ (\mathcal{U} \to X) \mapsto \mathcal{U} \end{array} \right]$$

Conversely: Category fibred on groupoids $\longmapsto$ Pseudo-functor.

In fact, use (1) + Axiom of choice: Choice $\forall\, V \to \mathcal{U} \in \mathcal{C}$, $\forall\, x \in \mathcal{X}(\mathcal{U})$

$$\begin{tikzcd} y \arrow[r, dashed] \arrow[d, mapsto] & x \arrow[d, mapsto] \\ V \arrow[r] & \mathcal{U} \end{tikzcd}$$

We denote $y := f^* x$.

Then every map $f : V \to \mathcal{U}$ defines a map on objects $f^* : \mathcal{X}(\mathcal{U}) \to \mathcal{X}(V)$, $x \mapsto f^* x$

Problem: If $g : W \to V$ in $\mathcal{C}$ then $g^*(f^*(x))$ may not be the same as $(f \circ g)^*(x)$. However, use axiom (2):

$$\begin{tikzcd} (f\circ g)^* x \arrow[d, "\exists!\,\varphi_2"'] \arrow[drr, bend left] \arrow[drrr, bend left, "\exists!\,\varphi_1"] \arrow[dd, mapsto, bend right] & & & \\ g^*(f^*(x)) \arrow[d, mapsto] & & f^* x \arrow[r] \arrow[d, mapsto] & x \arrow[d, mapsto] \\ W \arrow[rr, "g"'] & & V \arrow[r, "f"] & \mathcal{U} \end{tikzcd}$$

We get φ_2 which is a morphism in $\mathcal{X}$ over id_W

⇒ it is invertible, canonically defined.

⇝ it is the natural transformation (2-isomorphism) $\varepsilon_{f,g}(x)$ st $g^* \circ f^* \overset{\varepsilon_{f,g}}{\simeq} (f \circ g)^*$

Then, $\mathcal{U} \mapsto \mathcal{X}(\mathcal{U})$ + the $\varepsilon_{f,g}$ defines a pseudo-functor $\mathcal{C} \to \text{Grpds}$. ■

If $\mathcal{X}, \mathcal{Y}$ are CFG over $\mathcal{C}$,

- A morphism $F : \mathcal{X} \to \mathcal{Y}$ is a functor that commutes $\begin{tikzcd} \mathcal{X} \arrow[rr, "F"] \arrow[dr, "p_{\mathcal{X}}"'] & & \mathcal{Y} \arrow[dl, "p_{\mathcal{Y}}"] \\ & \mathcal{C} & \end{tikzcd}$
- A natural transformation between $F, G : \mathcal{X} \to \mathcal{Y}$ is $\begin{tikzcd} \mathcal{X} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\alpha \Downarrow"] & \mathcal{Y} \end{tikzcd}$ over the identity $id_{\mathcal{C}}$.

(⇝ we get a 2-category)

• <u>Fiber products</u> of $\mathcal{F}: \mathcal{X} \to \mathcal{Y}$ and $\mathcal{F}': \mathcal{X}' \to \mathcal{Y}$ is given by

Objects: $\{ (x, x', \alpha) \mid x \in \mathcal{X}(\mathcal{U}),\ x' \in \mathcal{X}'(\mathcal{U}) \text{ and } \alpha : \mathcal{F}(x) \xrightarrow{\sim} \mathcal{F}'(x') \text{ in } \mathcal{Y}(\mathcal{U}) \}$

So, CFGr over $\mathcal{C}$ <u>form a 2-category</u> with a <u>2-fiber product</u>.

<u>Problem</u>: Until now, we can take any site $\mathcal{C}_\tau$ and any grupoid G then the "constant functor" $\mathcal{U} \mapsto G$ (or product CFGr $\mathcal{C} \times G \to \mathcal{C}$) defines a stack.

<u>Algebraic stacks</u>:

Let $\mathcal{C}_\tau$ be the site $(Sch/S)_{ét}$. Let $\mathcal{X}$ be a stack.

<u>Recall</u>: $\mathcal{X}$ is representable $\iff \mathcal{X} \simeq \underline{X}$ for some $X \in \mathcal{C}$.

<u>Def</u>: $\mathcal{F}: \mathcal{X} \to \mathcal{Y}$ is <u>representable</u> $\iff \forall Y \in \mathcal{C},\ \forall \underline{Y} \xrightarrow{G} \mathcal{Y}$ then $\underline{Y} \times_{\mathcal{Y}} \mathcal{X}$ is <u>representable</u> (by some $Z \in \mathcal{C}$).

The composition, products and base change of representable morphisms are representable.

<u>General principle</u>: If $\mathbb{P}$ is a property of morphisms of schemes that is
- (stable by composition)
- local on the target [ie, $\forall f: X \to Y$ and every $\{U_i \to Y\}$ cover then f has $\mathbb{P} \iff \forall i\ X \times_Y U_i \to U_i$ has $\mathbb{P}$]
- stable by base change
  $f: X \to Y$ has $\mathbb{P} \Rightarrow \forall Z \to Y,\ X \times_Y Z \to Z$ has $\mathbb{P}$

<u>Examples</u> (of $\mathbb{P}$): étale, surjective, smooth, ...

Then, we say that a representable morphism $\mathcal{F}: \mathcal{X} \to \mathcal{Y}$ has $\mathbb{P}$ $\overset{def}{\iff} \forall \underline{Y} \to \mathcal{Y},\ \underline{Y} \times_{\mathcal{Y}} \mathcal{X} \to \underline{Y}$ has $\mathbb{P}$. (as morphism between schemes).

<u>Consequence</u>: The class of representable morphisms of stacks having property $\mathbb{P}$ is again stable by composition, base change.

Def : On $(Sch/S)_{ét}$, $\mathcal{X}$ is a Deligne - Mumford stack if :

1) The diagonal $\Delta : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is representable, quasi - compact and separated.

2) $\exists$ scheme X (called atlas) and a morphism $\underline{X} \to \mathcal{X}$ surjective and étale

If $\mathbb{P}$ is a property of schemes, we say that $\mathcal{X}$ has $\mathbb{P}$ if it has an atlas X which has $\mathbb{P}$.

Rmk : Property 1 will imply (+ hypothesis) : the group of automorphisms are finite .

Property : " $\mathcal{X}$ is a quotient of X "

Case of a quotient stack $[X/G]$ we can take X as an atlas .

Case of the stack of vector bundles / coherent sheaves $\rightsquigarrow$ Quot scheme .

Case of family of curves $\rightsquigarrow$ Hilbert scheme .