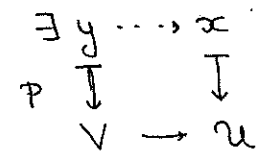


Lecture 4: "Fibered categories and algebraic stacks"

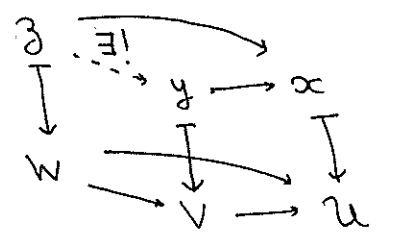
Fix  $\mathcal{C}$  any category.

Def: A category over  $\mathcal{C}$  is a category  $\mathcal{X}$  with a functor  $p: \mathcal{X} \rightarrow \mathcal{C}$ .  
 Then  $\forall U \in \mathcal{C}$  we consider  $\mathcal{X}(U)$  to be the category with objects over  $U$  and morphisms over  $\text{id}_U$ :  $x \in \mathcal{X}$  st  $p(x) = U$ ,  
 $f: x \rightarrow y$  st  $p(f) = \text{id}_U$ .

Def: A category fibered in groupoids is a category  $\mathcal{X}$  over  $\mathcal{C}$  st  
 ①  $\forall f: V \rightarrow U$  in  $\mathcal{C}$  and  $\forall x \in \mathcal{X}(U)$ ,  $\exists y \rightarrow x$  st  $p(y \rightarrow x) = f$ .



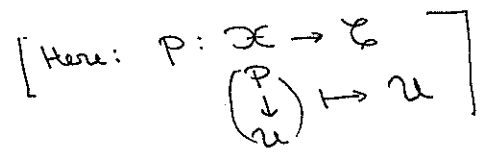
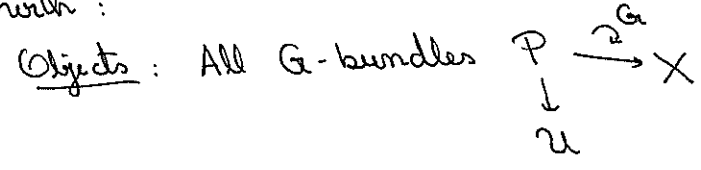
② Given



In this case,  $\forall U \mathcal{X}(U)$  is a groupoid.

Examples:

1) Consider the prestack  $[X/G]$ . We associate to it the category  $\mathcal{X}$  with:



Morphisms: Morphisms of principal bundles.

Then the axiom:

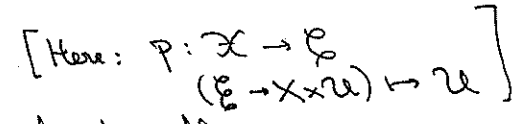
① says that  $\exists$  pull-back of principal bundles.

② Take  $W = V \rightarrow$  universal property of pull-back.

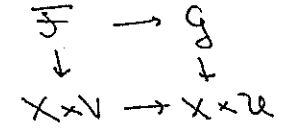
2) Prestack of vector bundles of rank  $n$  over  $X$ .

We associate to the category  $\mathcal{X}$  with

Objects: All vector bundles over  $X \times U$



Morphisms: Over  $V \rightarrow U$ : morphism of vector bundles



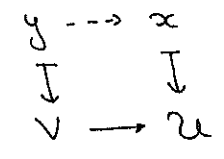
3) Prestack associated to an object  $X \in \mathcal{C} \rightsquigarrow$  We assoc. the category

$\mathcal{X} : (\mathcal{C}/X)^{\text{op}}$  with

Objects:  $U \rightarrow X$       [ Here:  $\varphi : \mathcal{X} \rightarrow \mathcal{C}$   
 $(U \rightarrow X) \mapsto U$  ]  
Morphisms (as usual)

Conversely: Category fibered on  $\mathcal{C} \mapsto$  Pseudo-functor.

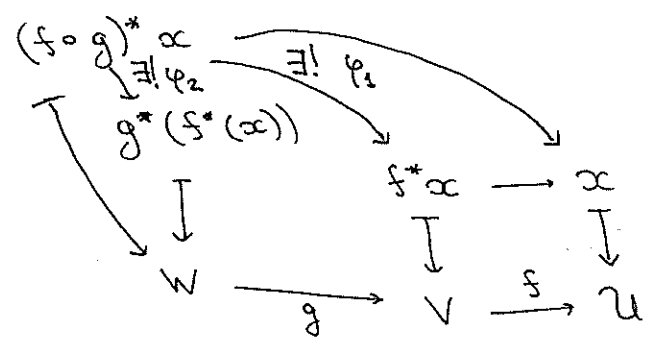
In fact, use ① + Axiom of choice: choice  $\forall U \rightarrow V \in \mathcal{C}$   
 $\forall x \in \mathcal{X}(U)$



We ~~denote~~ denote  $y := f^*x$ .

Then every map  $f : V \rightarrow U$  defines a map on objects  $f^* : \mathcal{X}(U) \rightarrow \mathcal{X}(V)$   
 $x \mapsto f^*x$

Problem: If  $g : W \rightarrow V$  in  $\mathcal{C}$  then  $g^*(f^*(x))$  may not be the same  
as  $(f \circ g)^*(x)$ . However, use axiom ②:



We get  $\varphi_2$  which is a morphism in  $\mathcal{X}$  over  $\text{id}_W$   
 $\Rightarrow$  it is invertible, canonically defined.

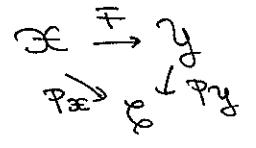
$\rightarrow$  it is the natural transformation (2-isomorphism)

$$E_{f,g}(x) \text{ st } g^* \circ f^* \xrightarrow{E_{f,g}} (f \circ g)^*$$

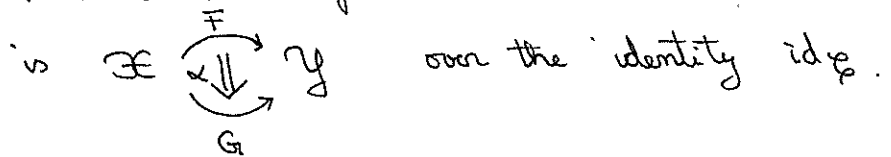
Then,  $U \mapsto \mathcal{X}(U)$  + the  $E_{f,g}$  defines a pseudo-functor  $\mathcal{C} \rightarrow \text{Grpds}$ . ■

If  $\mathcal{X}, \mathcal{Y}$  are CFG over  $\mathcal{C}$ ,

• A morphism  $F : \mathcal{X} \rightarrow \mathcal{Y}$  is a functor that commutes



• A natural transformation between  $F, G : \mathcal{X} \rightarrow \mathcal{Y}$



( $\rightsquigarrow$  we get a 2-category)

• Fiber products:  $F: \mathcal{X} \rightarrow \mathcal{Y}$  and  $F': \mathcal{X}' \rightarrow \mathcal{Y}$  is given by (11)

Objects:  $\{(x, x', \alpha) \mid x \in \mathcal{X}(U), x' \in \mathcal{X}'(U) \text{ and } \alpha: F(x) \xrightarrow{\sim} F'(x') \text{ in } \mathcal{Y}(U)\}$

So, CFGs over  $\mathcal{C}$  form a 2-category with a 2-fiber product.

Problem: Until now, we can take any site  $\mathcal{C}_\tau$  and any groupoid  $G$ . Then the "constant functor"  $U \mapsto G$  (or product CFG  $\mathcal{C} \times G \rightarrow \mathcal{C}$ ) defines a stack.

Algebraic stacks:

Let  $\mathcal{C}_\tau$  be the site  $(\text{Sch}/S)_{\text{ét}}$ . Let  $\mathcal{X}$  be a stack.

Recall:  $\mathcal{X}$  is representable  $\iff \mathcal{X} \simeq \underline{X}$  for some  $X \in \mathcal{C}$ .

Def:  $F: \mathcal{X} \rightarrow \mathcal{Y}$  is representable  $\iff \forall Y \in \mathcal{C}, \forall Y \xrightarrow{G} \mathcal{Y}$  then  $Y \times_{\mathcal{Y}} \mathcal{X}$  is representable (by some  $Z \in \mathcal{C}$ ).

The composition, products and base change of representable morphisms are representable.

General principle: If  $\mathbb{P}$  is a property of morphisms of schemes that is

- (stable by composition)

- local on the target [ie,  $\forall f: X \rightarrow Y$  and every  $\{U_i \rightarrow Y\}$  cover then  $f \text{ has } \mathbb{P} \iff \forall i, X \times_Y U_i \rightarrow U_i \text{ has } \mathbb{P}$ ]

- stable by base change

$f: X \rightarrow Y \text{ has } \mathbb{P} \Rightarrow \forall Z \rightarrow Y, X \times_Y Z \rightarrow Z \text{ has } \mathbb{P}$

Examples (of  $\mathbb{P}$ ): étale, surjective, smooth, ...

Then, we say that a representable morphism  $F: \mathcal{X} \rightarrow \mathcal{Y}$  has  $\mathbb{P}$

$\iff \forall Y \rightarrow \mathcal{Y}, Y \times_{\mathcal{Y}} \mathcal{X} \rightarrow Y$  has  $\mathbb{P}$ . (as morphism between schemes).

Consequence: The class of representable morphisms of stacks having property  $\mathbb{P}$  is again stable by composition, base change.

Def:  $\text{Om}(\text{Sch}/S)_{\text{ét}}$ ,  $\mathcal{X}$  is a Deligne - Mumford stack if:

- 1) The diagonal  $\Delta: \mathcal{X} \rightarrow \mathcal{X} \times \mathcal{X}$  is representable, quasi-compact and separated.
- 2)  $\exists$  scheme  $X$  (called atlas) and a morphism  $X \rightarrow \mathcal{X}$  surjective and étale

If  $\mathbb{P}$  is a property of schemes, we say that  $\mathcal{X}$  has  $\mathbb{P}$  if it has an atlas  $X$  which has  $\mathbb{P}$ .

Remark: Property 1 will imply (+ hypothesis): the group of automorphisms are finite.

Property: " $\mathcal{X}$  is a quotient of  $X$ "

Case of a quotient stack  $[X/G]$  we can take  $X$  as an atlas.

Case of the stack of vector bundles/coherent sheaves  $\rightarrow$  Quot scheme.

Case of family of curves  $\rightarrow$  Hilbert scheme.