Lecture 5: "The moduli stack of curves" / $k = \bar{k}$ (of any characteristic), $g \geq 2$

References: "The irreducibility of the space of curves of given genus" Deligne & Mumford
"Moduli of curves" Harris & Morrison
"Notes on the construction of the moduli space of curves" Edidin.

Def: Let S be a scheme$/k$. A smooth curve of genus $g \geq 2$ over S is a proper, smooth family $C \xrightarrow{\pi} S$ whose geom. fibers are smooth connected 1-dim schemes C_s such that $\dim H^0(C_s, \omega_{C_s}) = \dim H^1(C_s, \mathcal{O}_{C_s}) = g$.

Rmk: Here ω_{C_s} is the "dualizing sheaf" of regular 1-forms on C_s.

Consider the functor $\mathcal{F} : (Sch/k)^{op} \to Sets$

$$S \longmapsto \left\{ \begin{array}{c} \text{smooth curves} \\ \text{of genus } g \text{ over } S \end{array} \right\} / \text{isom}.$$

Rmk: $\mathcal{F}$ representable by M_g scheme$/k$ $\iff$ $\mathcal{F}(S) \simeq \mathrm{Hom}(S, M_g)$

In part, $\mathcal{F}(\mathrm{Spec}\, k) \simeq M_g(k)$ parametrize isom. classes of curves of genus g.

Problem 1: $\mathcal{F}$ not representable !

Let X a smooth curve$/k$ with $\mathrm{Aut}(X)$ finite and non-trivial

Eg. Let $n \geq 4$, $\mathrm{char}(k) \nmid n$ and let $X = \{ [x:y:z] \in \mathbb{P}^2(k) \mid x^n + y^n + z^n = 0 \}$
$\Rightarrow \mathrm{Aut}(X)$ finite non-trivial, $S_3 \ltimes (\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}) \hookrightarrow \mathrm{Aut}(X)$
Equality if $n \neq 1 + p^n$, $\mathrm{char}(k) = p > 0$ (see H.W. Leopoldt) or $\mathrm{char}(k) = 0$

Let B' scheme with free $G = \mathrm{Aut}(X)$ action and let $B = B'/G$, $C' = B' \times X$
$\Rightarrow G \curvearrowright C'$ and $C'/G := C \to B$ is a family of curves st
$C_b \simeq X$ for all $b \in B$, but in gral $C \not\simeq X \times B$ (cf. Beauville surfaces)
↳ The image of $B \to M_g$ would be a point $\Rightarrow C \simeq X \times B$.

Def: A scheme M_g is a coarse moduli space for $\mathcal{F}$ is there is a nat. transf. $\phi : \mathcal{F} \to \mathrm{Hom}(-, M_g)$ st

1) For any alg closed field Ω, $\phi : \mathcal{F}(\mathrm{Spec}\, \Omega) \xrightarrow{\sim} M_g(\Omega)$ bijection.
2) For any scheme S and $\psi : \mathcal{F} \to \mathrm{Hom}(-, S)$ nat. transf.
$\exists!\, \chi : \mathrm{Hom}(-, M_g) \to \mathrm{Hom}(-, S)$ nat. transf st $\psi = \chi \circ \phi$

Rmk: A map $S \to M_g$ does not induces a family over S in gral
But given $C \to S$ family $\rightsquigarrow \exists\, S \to M_g$.

Thm (GIT, Ch. 5): There exists M_g coarse moduli space of dim $3g - 3$ which is quasi-proj and irred$/k$.

Rmk: Deformation Theory: tangent space of $M_g(k)$ at $[C]$ has dimension

$\dim H^1(C, T_C) = h^1(C, \omega_C^\vee) = h^0(C, \omega_C^{\otimes 2})$

Riemann-Roch: $h^0(C, \omega_C^{\otimes 2}) = 2\cdot(2g-2)+1-g = 3g-3 \Rightarrow \dim M_g = 3g-3$

Problem 2: M_g is not complete !

Example $(g=3)$: Consider $\{x^4 + xyz^2 + y^4 + t(z^4 + z^3x + z^3y + z^2y^2) = 0\} \subseteq \mathbb{P}^2 \times \mathrm{Spec}\,\mathcal{O}$

where $\mathcal{O}$ D.V.R. with unif. parameter t.

Total space is smooth, but for "$t=0$" the fiber has a node in $[0:0:1] \in \mathbb{P}^2$. Moreover, after any base change and any blow-up there is always a (-2)-curve, so no modification gives a smooth family. (cf. Kollár-Mori Ch. 4)

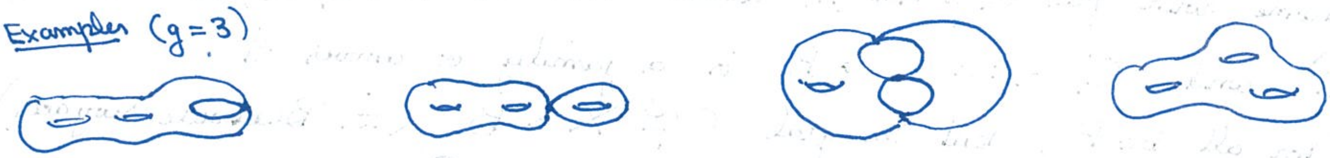
Def (DM) A stable curve of genus $g \geq 2$ over a scheme S is a proper flat family $C \xrightarrow{\pi} S$ whose geom. fibers are reduced, connected, 1-dim schemes C_s st:

1) C_s has at most ordinary double points as singularities (locally $k[x,y]/(xy)$ )

2) $\dim H^1(C_s, \mathcal{O}_{C_s}) = g$ (arithmetic genus)

3) Every non-sing rational component E of C_s meets the other comp. of C_s in ≥ 3 points.

Examples $(g=3)$



Rmk: $\mathrm{Aut}(\mathbb{P}^1) \simeq PGL_2$ 3-transitive

$\mathrm{Aut}(E)$, E elliptic curve 1-transitive

$\Rightarrow$ If C_s stable curve $\Rightarrow \mathrm{Aut}(C_s)$ is finite and reduced.

$\boxed{\text{Aim}}$: Use stable curves and stacks to construct M_g and its compactification $\overline{M_g}$.

Let us consider the categories fibred in groupoids over the site $(Sch/k)_{ét}$:

$$\mathcal{M}_g \longrightarrow Sch/k$$

and

$$\overline{\mathcal{M}}_g \longrightarrow Sch/k$$

whose objects (sections) over a scheme S are families of smooth (resp. stable) curves $C \to S$ (with morph. being isom. of schemes$/S$ $\rightsquigarrow$ groupoid!) and with morphisms from $C' \to S'$ to $C \to S$ given by the cartesian diagram

$$\begin{tikzcd} C' \arrow[r] \arrow[d] & C \arrow[d] \\ S' \arrow[r] & S \end{tikzcd}$$

<u>Theorem</u>: $\mathcal{M}_g$ and $\overline{\mathcal{M}}_g$ are DM stacks.

<u>Outline of the proof</u>:

1°) Some theory of duality
2°) Hilbert schemes
3°) $\mathcal{M}_g$ and $\widetilde{\mathcal{M}}_g$ are quotient stacks
4°) $\mathcal{M}_g$ and $\widetilde{\mathcal{M}}_g$ are DM stacks.

1° Recall that for Noetherian local rings (hence for Noeth. schemes) we have:

Cohen-Macaulay (CM) $\supseteq$ Gorenstein $\supseteq$ (Local) Complete intersections $\supseteq$ Regular ("smooth")

We can define a dualizing sheaf ω_S

ω_S is a line bundle

Nice deformation theory ($\Rightarrow \dim \overline{\mathcal{M}}_g = 3g-3$)

We can do the same in the relative setting (ie, for morphisms).

Let $\pi : C \to S$ be a stable curve of genus $g \geq 2$

$\Rightarrow \pi$ l.c.i morphism. In part, $\exists$ dualizing (invertible) sheaf $\omega_{C/S}$ on C.

<u>Example</u>: $S = \mathrm{Spec}(k)$ and C smooth: $\omega_C = \Omega_C = (T_C)^\vee$

$$\deg(\omega_C^{\otimes n}) = n(2g-2) \begin{cases} > 2g-1 & \text{if } n \geq 2 \Rightarrow H^1(C, \omega_C^{\otimes n}) = \{0\}, \ n \geq 2 \\ > 2g & \text{if } n \geq 3 \Rightarrow \omega_C^{\otimes n} \text{ very ample}, \ n \geq 3 \end{cases}$$

RR: $h^0(C, \omega_C^{\otimes n}) = n(2g-2) + 1 - g = (2n-1)(g-1)$.

<u>Claim</u> (DM p.78): Let $C \xrightarrow{\pi} S$ stable of genus $g \geq 2$
$\Rightarrow \omega_{C/S}^{\otimes n}$ relatively very ample for $n \geq 3$ and $\pi_*(\omega_{C/S}^{\otimes n})$ is a vector bundle (loc free sheaf) of rank $(2n-1)(g-1)$ on S.

② Hilbert polynomial (see Hartshorne Ch 1, §7):

Let $X^n \overset{i}{\hookrightarrow} \mathbb{P}^N_k$ projective variety and $\mathcal{O}_X(1) := i^*\mathcal{O}_{\mathbb{P}^N_k}(1)$ very ample.

$\Rightarrow \chi(X, \mathcal{O}_X(m)) := P_X(m)$ is a polynomial of deg $\dim X = n$

↳ the "Hilbert polynomial" of $(X, \mathcal{O}_X(1))$

Serre vanishing: For $m \gg 0$,

$$P_X(m) = \dim_k H^0(X, \mathcal{O}_X(m))$$

Main example: $\mathcal{L} = \omega_C^{\otimes n}$ over C smooth proj curve ($n \geq 3$ fixed).

$\Rightarrow \mathcal{L}$ very ample and defines $C \overset{i}{\hookrightarrow} \mathbb{P}^N_k$ with $N = (2n-1)(g-1)-1$

and $\mathcal{O}_X(1) := i^*(\mathcal{O}_{\mathbb{P}^N_k}(1)) = \mathcal{L}$.

$\Rightarrow P_C(m) \underset{m \gg 0}{=} h^0(C, \mathcal{L}^{\otimes m}) \underset{RR}{\overset{\uparrow}{=}} nm(2g-2)+1-g = (2nm-1)(g-1)$.

Relative setting: $\omega_{C/S}^{\otimes n}$ rel. very ample $\rightsquigarrow C \xrightarrow{\pi} S$ can be realized as a family of curves in $\mathbb{P}^N_S = \mathbb{P}^N_k \times S$, with Hilbert polynomial

$$P_{g,n}(t) = (2nt-1)(g-1).$$

Grothendieck: $\exists$ projective scheme H (the Hilbert scheme) parametrizing closed subschemes of $\mathbb{P}^N_k$ with Hilbert polynomial $P_{g,n}(t)$.

DM (cf. GIT Prop. 5.1): $\exists!$ subschemes $H_{g,n} \subseteq \overline{H_{g,n}} \subseteq H$ corresponding to n-canonically embedded stable (resp. smooth for $H_{g,n}$) curves.

More precisely:

$$\mathrm{Hom}(S, \overline{H_{g,n}}) \simeq \left\{\begin{array}{l} \text{set of stable curves } C \xrightarrow{\pi} S \text{, plus} \\ \text{isom. } \mathbb{P}(\pi_*(\omega_{C/S}^{\otimes n})) \simeq \mathbb{P}^N \times S \\ \text{(modulo isom.)} \end{array}\right\}$$

Clear: PGL_{N+1} naturally acts on $H_{g,n}$ and $\overline{H_{g,n}}$.

③ Let us prove now that

$$\mathcal{M}_g \simeq [H_{g,n}/PGL_{N+1}] \text{ and } \overline{\mathcal{M}}_g \simeq [\overline{H_{g,n}}/PGL_{N+1}].$$

We shall prove the stable case (smooth case is similar):

Recall: $[X/G]$ is the category fibered in groupoids over $(Sch/k)_{et}$ such that the sections over the scheme S are:

G-principal bundles $P \to S$ with $P \to X$ G-equivariant map

Consider the functor $\varphi : \overline{M}_g \longrightarrow [\overline{H}_{g,n}/PGL_{N+1}]$

$(C \xrightarrow{\pi} S) \mapsto (P \to S$, the principal PGL_{N+1}-bundle assoc. to the proj. vector bundle $\mathbb{P}(\pi_*(\omega_{C/S}^{\otimes n})) \longrightarrow S$
fiber $\mathbb{P}^N \circlearrowleft PGL_{N+1}$

Let $C_P := C \times_S P \xrightarrow{\pi'} P$ be the pullback family

$$\begin{array}{ccc} C\times_S P & \xrightarrow{\pi'} & P \\ \downarrow & & \downarrow \\ C & \xrightarrow{\pi} & S \end{array}$$

$\Rightarrow \mathbb{P}(\pi'_*(\omega_{C_P/P}^{\otimes n})) \simeq \mathbb{P}^N_k \times P$ trivial bundle

Then, there is a map $P \to \overline{H}_{g,n}$ which is PGL_{N+1} equivariant.

Rmk: If $\begin{array}{ccc} C' & \to & C \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ S' & \xrightarrow{\phi} & S \end{array}$ morphism in $\overline{M}_g$ $\Rightarrow \pi'_*(\omega_{C'/S'}) \simeq \phi^* \pi_*(\omega_{C/S}) \rightsquigarrow$ we get a morph. of PGL_{N+1} bundles $\begin{array}{ccc} P' & \to & P \\ \downarrow & & \downarrow \\ S & \to & S \end{array}$

Recall: φ equivalence $\Longleftrightarrow$ φ fully faithful and essentially surjective

(Sketch).

a) $\underline{\varphi \text{ faithful}}$: If C stable curve$/k$ $\Rightarrow \phi \in Aut(C)$ non-trivial induces an automorphism ϕ^* of $\mathbb{P}(H^0(C, \omega_C^{\otimes n})) = \mathbb{P}^N$
ϕ^* non-trivial since is non trivial on $C \subseteq \mathbb{P}^N \Rightarrow \varphi$ faithful

b) $\underline{\varphi \text{ full}}$: Let $\phi \in PGL_{N+1}$ non-trivial which leaves $C \hookrightarrow \mathbb{P}^N$ invariant $\Rightarrow \phi$ acts non-trivially on C (the fixed locus of ϕ is a linear subspace of $\mathbb{P}^N$) $\Rightarrow \varphi$ is full.

c) $\underline{\varphi \text{ essentially surjective}}$: Let $P \to S$ in $[\overline{H}_{g,n}/PGL_{N+1}]$.
Want: $\exists\, C \xrightarrow{\pi} S$ st $\varphi(C \to S) \simeq (P \to S)$.

By representability: $P \to \overline{H}_{g,n}$ corresp. to a family of stable curves $C_P \xrightarrow{\pi_P} P$ st $\mathbb{P}(\pi_{P,*}(\omega_{C_P/P}^{\otimes n})) \simeq \mathbb{P}^N_P \circlearrowleft PGL_{N+1}$

P/PGL_{N+1} is a scheme.

Descent theory: $C = C_P/PGL_{N+1}$ is a scheme
$\Rightarrow C_P \simeq C \times_S P$ and $P \to S$ isom to $\varphi(C \xrightarrow{\pi} S)$.

Conclusion: $\overline{M}_g \simeq [\overline{H}_{g,n}/PGL_{N+1}]$.

4° Thm (Edidin Thm 2.1, Cor 2.2 / DM Thm 4.21):
Let X/S Noetherian scheme of finite type and G/S smooth affine group scheme (of fin. type $/S$) with $G \curvearrowright X$ st the stabilizers of geom. points are finite and reduced. Then:
1) $[X/G]$ is a DM stack
2) Trivial stabilizers $\Rightarrow$ $[X/G]$ algebraic space.
3) $[X/G]$ separated $\Leftrightarrow$ the action is proper

We apply this to $X = \overline{H}_{g,n}$ and $G = PGL_{N+1}$:
Since stable curves over $k = \bar{k}$ have finite and reduced automorphism groups $\Rightarrow [\overline{H}_{g,n}/PGL_{N+1}] \simeq \overline{\mathcal{M}}_g$ is a DM stack ■

Further properties (see DM's article for details):
- $\mathcal{M}_g$ and $\overline{\mathcal{M}}_g$ are separated DM stacks.
- $\overline{\mathcal{M}}_g$ is proper over $\operatorname{Spec}\mathbb{Z}$. (Valuative criterion of properness + Stable reduction)
- $\mathcal{M}_g$ and $\overline{\mathcal{M}}_g$ are smooth over $\operatorname{Spec}\mathbb{Z}$
(this follows from the fact that $H_{g,n}$ and $\overline{H}_{g,n}$ are smooth over $\operatorname{Spec}\mathbb{Z}$).
- $\overline{\mathcal{M}}_g \setminus \mathcal{M}_g$ is a normal crossing divisor. (same holds for $H_{g,n}$ and $\overline{H}_{g,n}$)
- $\mathcal{M}_g$ and $\overline{\mathcal{M}}_g$ has irred. geom. fibers over $\operatorname{Spec}\mathbb{Z}$.

Remark on the quotient $\overline{H}_{g,n}/PGL_{N+1}$:

Def: The moduli space of a DM stack $\mathfrak{X}$ is a scheme M with a morphism $\mathfrak{X} \to \underline{M}$ st
1) $\forall$ alg closed field Ω, $\mathfrak{X}(\operatorname{Spec}\Omega) \simeq M(\Omega)$
2) If N is any scheme and $\mathfrak{X} \to \underline{N}$ is a morphisme
$\Rightarrow \exists!\ M \to N$ st
$$\begin{tikzcd} \mathfrak{X} \arrow[r] \arrow[rr, bend right] & \underline{M} \arrow[r] & \underline{N} \end{tikzcd}$$

Rmk: If $\mathfrak{X}$ DM stack then $\mathfrak{X}$ may not have a moduli scheme.
Keel - Mori (1997): Always exists a moduli algebraic space!

GIT (Edidin Prop 4.2): If M is the GIT quotient of the scheme X by G with stab. of geom. points finite and reduced
$\Rightarrow M$ is the moduli space of the stack $[X/G]$.

Hence: $\overline{M}_g := \overline{H}_{g,n} /\!/ PGL_{N+1}$ do the work ■