Lecture 6: "Orbifolds"

Preliminaries:

A category $\mathcal{C}$ can be given either as

- Set of objects $\mathcal{C}_0$ + Hom-sets $\mathrm{Hom}(x,y)$
- Set of objects $\mathcal{C}_0$ + set of morphisms $\mathcal{C}_1$
  + maps $s,t : \mathcal{C}_1 \to \mathcal{C}_0$ (source, target) + map $\mathrm{id} : \mathcal{C}_0 \to \mathcal{C}_1$

If a group G acts on a set X, then $[X/G]$: quotient groupoid with

Objects $X_0 = X$

$\mathrm{Hom}(x,y) = \{ g \in G \mid g\cdot x = y \}$

or

$x \sim g\cdot x$

or

$X_1 = X \times G$

$s(x,g) = x$

$t(x,g) = g\cdot x$

$\mathrm{id}(x) = (x, 1_G)$

In this case, $\pi_0([X/G]) = X/G$ isomorphism classes of a groupoid (quotient set).

Manifolds

A manifold can be given as: Open sets $\mathcal{U}_i \subseteq \mathbb{R}^n$ + $\mathcal{U}_{ij} \subseteq \mathcal{U}_i$
+ $\varphi_{ij} : \mathcal{U}_{ij} \xrightarrow{\sim} \mathcal{U}_{ji}$ diffeomorphism (st cocycle condition).

To this we associate a groupoid X with

$$X_0 = \coprod \mathcal{U}_i$$

+ relations: $\underbrace{x}_{\in \mathcal{U}_{ij}} \sim \underbrace{\varphi_{ij}(x)}_{\in \mathcal{U}_{ji}}$

or: $X_1 = \coprod \mathcal{U}_{ij}$

$$\begin{tikzcd} & \coprod \mathcal{U}_{ij} \arrow[dl, "s"'] \arrow[dr, "t = \varphi_{ij}"] & \\ \mathcal{U}_i & & \mathcal{U}_j \end{tikzcd}$$

then $\pi_0(X) = \left(\coprod \mathcal{U}_i\right) / x \sim \varphi_{ij}(x)$ underlying topological space.

Orbifolds

An orbifold is described as: Open sets $\mathcal{U}_i \subseteq \mathbb{R}^n$

+ Finite groups $G_i \curvearrowright \mathcal{U}_i$ (acts by diffeomorphism)

+ Open sets $\mathcal{U}_{ij} \subseteq \mathcal{U}_i$

$$G_{ij} = \{ g \in G_i \mid g(\mathcal{U}_{ij}) \subseteq \mathcal{U}_{ij} \}$$

+ $\psi_{ij} : G_{ij} \xrightarrow{\sim} G_{ji}$ group morphism

$\varphi_{ij} : \mathcal{U}_{ij} \xrightarrow{\sim} \mathcal{U}_{ji}$ equivariant:

$$\varphi_{ij}(g \cdot x) = \psi_{ij}(g) \cdot \varphi_{ij}(x)$$

To this data we associate a groupoid X with:

$$X_0 = \coprod \mathcal{U}_i$$

+ morphisms:

$$x \sim \varphi_{ij}(x)$$
$$x \sim g \cdot x$$

In this case $X_1 = \coprod \mathcal{U}_{ij} \times G_{ij}$

Locally: $X \approx [\mathcal{U}_i / G_i]$

Def: X is a Lie groupoid if:

X_0, X_1 are differential manifolds

$s, t : X_1 \to X_0$ are smooth submersions. ($\Rightarrow X_1 \times_{X_0} X_1$ is a manifold)

$\mathrm{id} : X_0 \to X_1$ smooth

"Internal category in Diff"

category of ~~diam~~ differentiable manifolds

So Orbifolds $\longleftrightarrow$ Lie groupoids such that s, t are étale (= local diffeomorphism)

+ (s,t) is proper.

Question: What is a morphism of orbifolds?
When are two orbifolds equivalent?

Solution 1: Do everything with local charts
- Can describe a morphism ✓
But: Very restrictive
- No clear when they are equivalent.

Solution 2: Internal functor / equivalences.

Ex: Internal functor between Lie groupoids $X \to Y$:
smooth maps $X_0 \to Y_0$ that is a functor.
$X_1 \to Y_1$

Clearly we want: Internal functor of Lie groupoids $\Rightarrow$ Morphism of orbifolds
Internal equivalence $\Rightarrow$ "same" orbifold.

Problem: too restrictive!

Solution 3: Differentiable stacks

Diff is a site! : Cover of M: $\{\mathcal{U}_i \xrightarrow{\varphi_i} M\}$ st $\bigcup \varphi_i(\mathcal{U}_i) = M$.
We work with stacks over Diff.

Def: A differentiable stack is a stack $\mathfrak{X}$ over Diff such that
$\exists$ atlas $\underline{X} \to \mathfrak{X}$ (ie. representable morphism) smooth, surjective submersion

($\Rightarrow \forall\, \mathcal{U} \in$ Diff, $\underline{X} \times_{\mathfrak{X}} \underline{\mathcal{U}} \cong \underline{V}$ for V manifold and
$\underline{X} \times_{\mathfrak{X}} \underline{\mathcal{U}} \to \underline{\mathcal{U}}$ is induced by $V \to \mathcal{U}$ which is smooth, surjective submersion)

Example:
1) If X manifold $\to \underline{X}$ differentiable stack with atlas X.
2) If $G \curvearrowright X \to [X/G]$ given by

$$\begin{array}{ccc} P & \xrightarrow{G\text{-equiv}} & X \\ \downarrow{\scriptstyle \text{principal } G\text{-bundle}} & & \\ \mathcal{U} & & \end{array}$$

$$\begin{array}{c} I \\ \downarrow \\ \mathcal{U} \end{array}$$

Atlas: X

If X_0, X_1 is a Lie groupoid, we associate a differentiable stack $[X_0/X_1]$ as follows:

$\mathfrak{X} = [X_0/X_1]$

Objects: $\begin{array}{c} P \\ q \downarrow \\ \mathcal{U} \end{array}$ surjective submersion

$+ \varphi : P \to X_0$

$+ \rho : P \underset{\varphi, X_0, s}{\times} X_1 \to P$

such that:

1) $q(\rho(p,g)) = q(p) \qquad p \in P, \; g \in X_1$

2) $\forall p, p' \in P \;, \; q(p) = q(p')$

$\Rightarrow \exists! \; g \in X_1 \;, \; p' = \rho(p,g)$

3) $\varphi(\rho(p,g)) = t(g) \qquad , g \in X_1, \; p \in P$

Rmk: In case of $[X/G]$: $\rho : G \curvearrowright P$

1): actions preserves the fibers

2): actions transitive on fibers

$\varphi : P \to X$

3): G - equivariant

$\Rightarrow [X_0/X_1]$: differentiable stack with atlas X_0.

Reverse direction: Let $\mathfrak{X}$ be a differentiable stack with atlas $\underline{X}$.

$$\begin{tikzcd} \underline{V} \cong \underline{X} \times_{\mathfrak{X}} \underline{X} \arrow[r] \arrow[d] & \underline{X} \arrow[d] \\ \underline{X} \arrow[r] & \mathfrak{X} \end{tikzcd}$$

we take $X_0 = X$

$X_1 = V$ with $s, t : X_1 \to X$

$\Rightarrow X_0, X_1$ is a Lie groupoid.

<u>Def 1</u> : An <u>orbifold</u> is a differentiable stack $\mathfrak{X}$ such that the associated Lie groupoid has s, t étale and (s,t) proper.

<u>Def 2</u> : An orbifold is a DM stack, ie the atlas $\underline{X} \to \mathfrak{X}$ smooth, surjective, étale ($\Rightarrow s, t$ étale) & the diagonal $\Delta : \mathfrak{X} \to \mathfrak{X} \times \mathfrak{X}$ is proper.

If X is a Lie groupoid, then $\pi_0(X)$ topological space is the "coarse moduli space":
$\forall$ manifold M, every morphism $X \to M$ factors through $\pi_0(X)$.

If $\mathfrak{X}$ is a stack over Diff, then

$$\mathcal{U} \longmapsto \pi_0(\mathfrak{X}(\mathcal{U}))$$

is a sheaf over Diff.

Case when $\mathfrak{X}$ DM stack : this sheaf is associated to a manifold $\rightsquigarrow$ "the coarse moduli space".