Seminar of Deformation Theory

1. Flat morphisms : "continuous family of schemes"

Def: Let $A \in \underline{\text{Ring}}$, $M \in A\text{-}\underline{\text{Mod}}$. We say that M is flat over A if the functor

$$M \otimes_A - : A\text{-}\underline{\text{Mod}} \to A\text{-}\underline{\text{Mod}}$$
$$N \mapsto M \otimes_A N$$

is left exact, ie,

$0 \to N_1 \to N_2 \to N_3 \to 0$ exact

$\Rightarrow 0 \to M \otimes_A N_1 \to M \otimes_A N_2 \to M \otimes_A N_3 \to 0$ exact

~~Properties~~: if $f \in \underline{\text{Ring}}(A, B)$, then we say that B is flat over A if it is flat as a module.

Properties:

1) $M \in A\text{-}\underline{\text{Mod}}$ flat $\iff \forall \mathfrak{a} \subseteq A$ fin. gen, $\mathfrak{a} \otimes M \to M$ is injective

2) Base change: if $M \in A\text{-}\underline{\text{Mod}}$ is flat, $f: A \to B$ homo., then $M \otimes_A B$ is a flat B-module.

3) Transitivity: B flat A-alg, N flat B-module $\Rightarrow N$ flat A-module

4) Localization: $M \in A\text{-}\underline{\text{Mod}}$ is flat $\iff M_{\mathfrak{p}}$ is flat over $A_{\mathfrak{p}}$, $\forall \mathfrak{p} \in \operatorname{Spec} A$

5) $0 \to M' \to M \to M'' \to 0$ exact

M', M'' flat $\Rightarrow M$ flat

M, M'' flat $\Rightarrow M'$ flat

6) If A local noeth ring, M fin gen. A-Mod, then M flat $\iff M$ free

Def: Let $f : X \to Y$ morphism of schemes, $\mathcal{F} \in \mathcal{O}_X\text{-}\underline{\mathrm{Mod}}$. $\mathcal{F}$ is <u>flat over Y at $x \in X$</u> if $\mathcal{F}_x$ is a flat $\mathcal{O}_{f(x),Y}$-module

(via the natural map $f^{\#} : \mathcal{O}_{f(x),Y} \to \mathcal{O}_{x,X}$)

$\mathcal{F}$ <u>flat</u> over Y if it is flat $\forall x \in X$

X is <u>flat over Y</u> if $\mathcal{O}_X$ is.

We have analogue properties in the context of schemes (open immersions, base change, ...)

<u>Ex:</u> X noeth. scheme, $\mathcal{F} \in \underline{\mathrm{Coh}}(X)$, then

$\mathcal{F}$ flat over $X \iff \mathcal{F}$ locally free.

<u>Prop:</u> Let X, Y scheme of fin. type $/ k$ and $f : X \to Y$ flat morphism. Let $y = f(x)$, for $x \in X$. Then,

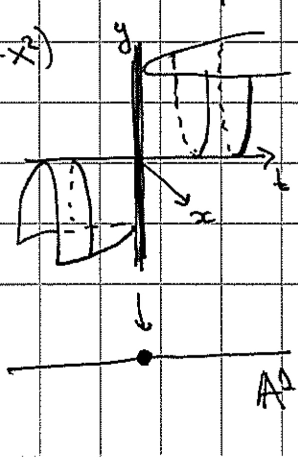
$$\dim_x (X_y) = \dim_x X - \dim_y Y$$

(Here $\dim_x X = \dim_{\text{Krull}} \mathcal{O}_{x,X}$)

<u>Example:</u> $\varepsilon : \tilde{X} \to X$ blow-up of a smooth subvariety of X (smooth), then ε is <u>not flat</u>.

<u>Important:</u> The properties "irreducible" and "reduced" are <u>not</u> preserved ~~by~~ in flat families: $k = \bar{k}$, char $k \neq 2$

$$\begin{array}{c} \operatorname{Spec} k[x,y,t]/(ty - x^2) \\ \downarrow f \\ \operatorname{Spec} k[t] \end{array}$$



$$\begin{array}{c} \operatorname{Spec} k[x,y,t]/(xy - t) \\ \downarrow f \\ \operatorname{Spec} k[t] \ni a \end{array}$$

X_a hyperbola (red) $a \neq 0$

X_0 : $\times$ 2 lines.

So $\dim_x(X_y)$ is preserved in flat families, but what else?

2. Hilbert polynomials

Let k be a field, X projective scheme over k and $\mathcal{O}_X(1)$ an ample Cartier divisor, then

$$m \longmapsto \chi(X, \mathcal{O}_X(m))$$

is polynomial of degree $\dim X$, i.e., $\exists! P \in \mathbb{Q}[t]$ $\deg P = \dim X$, st $P(m) = \chi(X, \mathcal{O}_X(m))$ $\forall m \in \mathbb{Z}$

Rmk: $\mathcal{O}_X(1)$ ample $\Rightarrow$ $H^i(X, \mathcal{O}_X(m)) = 0$ $\forall m \gg 0$ and all $i > 0$

So for $m \gg 0$, $P(m) = \dim_k H^0(X, \mathcal{O}_X(m))$

P is called the Hilbert polynomial of X w.r.t. $\mathcal{O}_X(1)$.

Ex: $C \subseteq \mathbb{P}^n_k$ smooth curve of genus g and degree d

$\boxed{\mathcal{O}_C(1) := i^*\mathcal{O}_{\mathbb{P}^n}(1)}$ RR: $h^0(\mathcal{O}_C(m)) = h^0(\omega_C \otimes \mathcal{O}_C(-m)) + \underbrace{\deg \mathcal{O}_C(m)}_{dm} - g + 1$

For $m > 2g-2$, $h^0(\mathcal{O}_C(m)) = dm - g + 1$

$\Rightarrow P(t) = dt - g + 1$

Ex: $H_d \subseteq \mathbb{P}^n$ hypersurface

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-H_d) \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_{H_d} \to 0$$
$$\mathcal{O}_{\mathbb{P}^n}(-H_d) \simeq \mathcal{O}_{\mathbb{P}^n}(-d)$$

$\Rightarrow P(t) = \binom{t+n}{n} - \binom{t-d+n}{n}$ ~~pol de Hilb~~

Hilb. pol. of H_d

Because, $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ exact

$\Rightarrow \chi(\mathcal{G}) = \chi(\mathcal{F}) + \chi(\mathcal{H})$

* Matsumura 23.1

Theorem: Let T be an integral noeth scheme, $X \subseteq \mathbb{P}^n_T$ closed subscheme, P_t Hilbert pol. of $X_t \subseteq \mathbb{P}^n_{k(t)}$. Then, X is flat over T iff P_t is independent of t.

Proof (sketch): Replace $\mathcal{O}_X$ by any $\mathcal{F} \in \mathrm{Coh}(\mathbb{P}^n_T)$ to assume $X = \mathbb{P}^n_T$.

The question is local on T: $T = \mathrm{Spec}\,(\overset{\text{noeth}}{A}, \mathfrak{m})$

Are equivalent:

(i) $\mathcal{F}$ flat over T

(ii) $H^0(X, \mathcal{F}(m))$ free A-mod. of finite rank, $\forall m \gg 0$

(iii) Hilb pol. P_t of $\mathcal{F}_t$ on $X_t = \mathbb{P}^n_{k(t)}$ is indep of t.

$\boxed{(i) \Rightarrow (ii)}$ $H^i(X, \mathcal{F}(m)) = h^i(\check{C}^\bullet(\mathcal{U}, \mathcal{F}(m)))$

$\check{C}^i(\mathcal{U}, \mathcal{F}(m))$ is flat A-mod + $H^i(X, \mathcal{F}(m)) = 0$ for $i > 0$ for $m \gg 0$

$$\Rightarrow \quad 0 \to H^0(X, \mathcal{F}(m)) \to \underbrace{C^0(\mathcal{U}, \mathcal{F}(m))}_{\text{flat } A\text{-mod}} \to \dots \to \underbrace{C^n(\mathcal{U}, \mathcal{F}(m))}_{\text{flat } A\text{-mod}} \to 0$$

$\Rightarrow H^0(X, \mathcal{F}(m))$ flat and fin. gen $\Rightarrow$ free of fin rank ✓

$\boxed{(ii) \Rightarrow (i)}$ Let $S = A[x_0, \dots, x_n]$, $M = \bigoplus_{m \geq m_0} \underbrace{H^0(X, \mathcal{F}(m))}_{\text{free}}$

$\Rightarrow \mathcal{F} = \widetilde{M}$

M free (and hence flat) A-mod $\Rightarrow \mathcal{F}$ flat over A.

$\boxed{(ii) \Leftrightarrow (iii)}$ To prove ($\Rightarrow$): $P_t(m) = \mathrm{rank}_A H^0(X, \mathcal{F}(m))$ $\forall m \gg 0$

Let $t \in T \rightsquigarrow \mathfrak{p} \in \mathrm{Spec}\,A$, $T' = \mathrm{Spec}\,A_{\mathfrak{p}} \to T$

By base change w.c.s. t is the closed point of T

$X_t = X_0$, $\mathcal{F}_t = \mathcal{F}_0$, $k(t) = k$. Take a presentation of k over A:

$$A^q \to A \to k \to 0 \qquad (*)$$

$$\Rightarrow \quad \mathcal{F}^q \to \mathcal{F} \to \mathcal{F}_0 \to 0$$

$$\forall m \gg 0 \Rightarrow \quad H^0(X, \mathcal{F}(m)^q) \to H^0(X, \mathcal{F}(m)) \to H^0(X_0, \mathcal{F}_0(m)) \to 0$$

Tensoring $(*) \otimes_A H^0(X, \mathcal{F}(m))$ and comparing:

$$H^0(X_0, \mathcal{F}_0(m)) \cong H^0(X, \mathcal{F}(m)) \otimes_A k \quad \forall m \gg 0$$

$\Rightarrow \forall t \in T$, $H^0(X_t, \mathcal{F}_t(m)) \cong H^0(X, \mathcal{F}(m)) \otimes_A k(t)$ $\forall m \gg 0$

($\Leftarrow$) Known: We can check freeness of $H^0(X, \mathcal{F}(m))$ by comparing its rank at gen. point and closed pt of T.

Corollary: $\dim X_t = \deg P_t$, $p_a(X_t) = (-1)^r (P_t(0) - 1)$ and $\deg X_t = (r!)$ (lead. coeff. of P_t) are constant in flat families.

Thm. (Matsumura 23.1): Let $f: X \to Y$ be a proper morphism of k-schemes. Assume Y regular, X Cohen-Macaulay, $\dim X_y = \dim X - \dim Y$ $\forall y$ $\Rightarrow$ f is flat.

3. Hilbert schemes

Goal: Describe families of closed subschemes of a given scheme

Theorem: Let k be a field and $Y \subseteq X = \mathbb{P}^n_k$ be a closed subscheme. Then:

(a) $\exists$ $\mathcal{H}$ projective scheme, parametrizing closed subschemes of X with same Hilbert pol. P as Y, and $\exists$ $W \subseteq X \times \mathcal{H}$ universal subscheme, flat over $\mathcal{H}$ st the fibers of W over closed points $h \in \mathcal{H}$ are all closed subschemes of X with Hilbert pol P.

$\boxed{\mathcal{H} = \mathrm{Hilb}^P(X)}$

Furthermore, $\mathcal{H}$ is universal: if T is an scheme and $W' \subseteq X \times T$ is a closed subscheme, flat over T, all whose fibers are subschemes of X with Hilbert pol. P then $\exists!$ morphism $\varphi: T \to \mathcal{H}$ st $W' = W \times_{\mathcal{H}} T$ as subschemes of $X \times T$.

(b) If $y \in H$ corresponds to $Y \subseteq X$, then
$T_y \mathcal{H} \cong H^0(Y, N_{Y/X})$, where $N_{Y/X} = \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{I}_Y / \mathcal{I}_Y^2, \mathcal{O}_Y)$

(c) If Y is l.c.i and $H^1(Y, N_{Y/X}) = 0$, then $\mathcal{H}$ is smooth at y, of dimension $h^0(Y, N_{Y/X})$.

(d) If Y is l.c.i, the dimension of $\mathcal{H}$ at y is at least $h^0(Y, N_{Y/X}) - h^1(Y, N_{Y/X})$.

__Remark__: $\mathrm{Hilb}^P(X)$ represents the functor

$$\mathcal{F} = \mathrm{Hilb}^P(X) : \underline{\mathrm{Sch}} \longrightarrow \underline{\mathrm{Sets}}$$

$$\mathcal{F}(Z) = \mathrm{Hilb}^P(X)(Z) = \left\{ \begin{array}{l} \text{subschemes } V \subseteq X \times Z \\ \text{which are proper and flat over } Z \\ \text{and have Hilbert pol. } P \text{ in each fiber} \end{array} \right\}$$

in the sense that $\mathcal{F}(Z) \cong \mathrm{Hom}(Z, \mathrm{Hilb}^P(X))$.

$\boxed{\underline{\text{Idea of (a):}}}$

$$\{\text{subschemes of } \mathbb{P}^n_k\} \hookrightarrow \{\text{sub. vector spaces of } k[x_0, \dots, x_n]\}$$

Idea $\rightarrow$ (i) Replace the right-hand by a finite dim. Grass.

(*) (ii) The image is a subvariety

Let I_Y the ideal sheaf of $Y \subseteq \mathbb{P}^n_k$, ie,

$$0 \to I_Y \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_Y \to 0$$

__Thm:__ $\forall P \in \mathbb{Q}[t]$, $\exists N(P) \in \mathbb{Z}_{\geq 1}$ st if $I \subseteq \mathcal{O}_{\mathbb{P}^n_k}$ is and ideal with Hilbert pol. P, then $\forall n \geq N(P)$:

a) $h^i(\mathbb{P}^n, I(n)) = 0$, for $i \geq 1$

b) $I(n)$ globally generated

c) $H^0(\mathbb{P}^n, I(n)) \otimes H^0(\mathbb{P}^n, \mathcal{O}(1)) \to H^0(\mathbb{P}^n, I(n+1))$ is surjective.

So let N as in the Thm above, then

$$H^0(\mathbb{P}^n_k, \mathcal{I}_Y(N)) \subseteq H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(N))$$

determines Y since

$$\mathcal{I}_Y(N) = \mathrm{im}\left(\mathcal{O}_{\mathbb{P}^n} \otimes H^0(\mathbb{P}^n, \mathcal{I}_Y(N)) \hookrightarrow \mathcal{O}_{\mathbb{P}^n} \otimes H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(N)) \to \mathcal{O}_{\mathbb{P}^n}(N)\right)$$

Thm above $\Rightarrow$ N work $\forall\, Y \subseteq \mathbb{P}^n_k$ with the same Hilbert polynomial $P(t) \in \mathbb{Q}[t]$.

~~Remark~~ $\cancel{P(N) = \chi(Y, \mathcal{O}_Y(N))}$

$$\cancel{\to H^i(\mathbb{P}^n, \mathcal{I}_Y(N)) \to H^i(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(N)) \to H^i(Y, \mathcal{O}_Y(N)) \to \cdots}$$

~~For~~ $\cancel{N \gg 0 \quad P(N) = h^0(}$

$\mathcal{I}_Y \quad Q(n) = \chi(\mathbb{P}^n_k, \mathcal{I}_Y(n))$

Then, $Q(N) = h^0(\mathbb{P}^n_k, \mathcal{I}_Y(N))$

So we obtain an injective map

$$\left\{\begin{array}{l}\text{Subschemes of } \mathbb{P}^n \text{ with} \\ \text{Hilbert polynomial } P\end{array}\right\} \hookrightarrow \mathrm{Grass}(Q(N), H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(N)))$$

Remark. $\mathcal{U}$ is obtained as the universal object corresponding to $\mathrm{id}_{\mathrm{Hilb}^P(X)}$:

$$\begin{array}{ccc} \mathrm{id}_{\mathrm{Hilb}^P(X)} & \in & \mathrm{Hom}_{\underline{\mathrm{Sch}}}(\mathrm{Hilb}^P(X), \mathrm{Hilb}^P(X)) \\ \updownarrow & & \wr\wr \\ \mathcal{U} & \in & F(\mathrm{Hilb}^P(X)) \end{array}$$

Exemples :

i) $\mathrm{Hilb}^1(X) \cong X$

ii) $\mathrm{Hilb}^P(X \times_S Z / Z) \cong \mathrm{Hilb}(X/S) \times_S Z$
pour $Z \to S$.