Deformations over the dual numbers

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Recall on D

k: field $\quad D := k[t]/t^2$.

D is a finite k-algebra. Its maximal ideal is $m = (t)$ and residue field k: the projection is k-linear.

$D \to k$

$a \mapsto a \bmod m$

is the same as ~~$a = (x+0)$~~

$a = (x + ty) \mapsto x$.

We call it <u>reduction</u> ~~mod~~ <u>mod t</u>.

Dually: as scheme, D is over k and has a unique point over k, dual to the <u>reduction mod t</u>.

For any scheme X over k and $x: k \to X$ a k-rational point (ie x:
$$\begin{tikzcd} \mathrm{Spec}(k) \arrow[r] \arrow[d, "id"'] & X \arrow[dl] \\ \mathrm{Spec}(k) & \end{tikzcd}$$
)

the set of morphisms $D \to X$ such that
$$\begin{tikzcd} & k \arrow[dl] \arrow[dr, "x"] & \\ D \arrow[rr] & & X \end{tikzcd}$$
(over k)

is the <u>Zariski tangent space</u> of X at x.

Recall on Tor

A: ring $\quad M$: A-module.

There exists functors $\mathrm{Tor}_n^A(M, -)$ such that for every short exact sequence $0 \to N' \to N \to N'' \to 0$ there is a long exact sequence

$$\cdots \to \mathrm{Tor}_2^A(M, N'') \to \mathrm{Tor}_1^A(M, N') \to \mathrm{Tor}_1^A(M, N) \to \mathrm{Tor}_1^A(M, N'')$$

$$\to \mathrm{Tor}_0^A(M, N') \to \mathrm{Tor}_0^A(M, N) \to \mathrm{Tor}_0^A(M, N'') \to 0.$$

Properties: · $\mathrm{Tor}_0^A(M, N) = M \otimes_A N$

- $\mathrm{Tor}_n^A(M, -)$ is a covariant functor $A\text{-Mod} \to A\text{-Mod}$.
- In fact $\mathrm{Tor}_n^A(-, -)$ is a bifunctor and

$$\mathrm{Tor}_n^A(M, N) \simeq \mathrm{Tor}_n^A(N, M) \quad \left(\text{easy spectral sequence arguments}\right)$$

Clearly: M is flat over A $\Leftrightarrow$
$\forall N$ A-module $\mathrm{Tor}_1^A(M, N) = 0$.

Lemmas on flatness

Our goal is to study how to pass from flatness over $\cancel{D}$ k to flatness over D, so we need general lemmas

Lemma 2.1 A: noetherian. For an A-module M: M is flat $\Leftrightarrow$ for all prime ideal $\mathfrak{p} \subset A$ $\mathrm{Tor}_1^A(M, A/\mathfrak{p}) = 0$.

$\Rightarrow$: trivial

Proof. first step: reduce to the case of modules of finite type.

- every module N is a direct limit of finite type modules N_i, $i \in I$ (I: directed set $\forall a, b \in I$ $\exists c \in I$ $a \leq c$ and $b \leq c$)

this is almost tautological:
$I = \{\text{finite subsets of } N\}$, N_i = submodule generated by i.

- $\mathrm{Tor}^A(M, -)$ commutes with direct limits.

In fact: first show that $M \otimes_A -$ does:

basically a direct limit is a cokernel

$$Q \to \bigoplus_{j \in I} N_j \to \varinjlim N_j \to 0$$

Q : relation "if $i \leq k$, $j \leq k$ $x_i = x_j$ in N_k"

and $M \otimes_A$ commutes with $\oplus$ and is right exact.

This is then a very general argument: the colimit functor is exact and commutes with the tensor product functor so it is easy to see that it commutes with the derived functors, which are the $\mathrm{Tor}^A(M, -)$.

- second step: if N is of finite type over a noetherian ring A, then $\exists\ 0 \subset N_1 \subset N_2 \subset \dots \subset N_m = N$ such that $N_{i+1}/N_i \simeq A/p_i$ p_i : prime ideal of A.

(idea: N is a noetherian module; consider N' a maximal submodule in {sub-modules which admit such a filtration}; show $N' = N$ by considering $N'' := N/N'$)

then: use induction on the length of the filtration.

~~$m = 1$: $N_1 = A/p_1 \to$ hypothesis~~

to show that $\mathrm{Tor}_1^A(M, N) = 0$ for every N.

$m = 1$: $N = A/p_0 \to$ hypothesis

$m = 2$: $0 \subset N_1 \subset N_2 = N$ hypothesis: $\mathrm{Tor}_1^A(M, N_1) = 0$

$N_2/N_1 = A/p_1 \to$

write the short exact sequence $0 \to N_1 \to N_2 \to N_2/N_1 \to 0$

then $\to \underbrace{\mathrm{Tor}_1^A(M, N_1)}_{=0} \to \mathrm{Tor}_1^A(M, N_2) \to \underbrace{\mathrm{Tor}_1^A(N_2/N_1, M)}_{=0} \to$

so $\mathrm{Tor}_1^A(M, N_2) = 0$

Proposition 2.2 If $A' \to A$ is a surjective morphism of noetherian rings whose kernel J has square zero (ex. $A' = D$, $A = k$). An A'-module M' is flat over A $\Leftrightarrow$

1. $M := M' \otimes_{A'} A$ is flat over A and
2. the natural map $M \otimes_A J \to M'$ is injective.

Proof: remark: $J^2 = 0 \Rightarrow J$: A-module $(A = A'/J)$
and $M' \otimes_{A'} J = M \otimes_A J$

$\Rightarrow$: easy 1. base change
2. $=$ natural map $M' \otimes_{A'} J \to M'$
known!

$\Leftarrow$: Let $\mathfrak{p}'$: prime ideal of A'. (~~but~~ goal: $\mathrm{Tor}_1^{A'}(M', A'/\mathfrak{p}') = 0$)
J nilpotent $\Rightarrow$ $J \subset \mathfrak{p}'$.
~~Let $\mathfrak{p} = \pi^{-1}(\mathfrak{p}')$ $\pi: A' \to A'/J$ $(\simeq A)$~~
so $\mathfrak{p}' = \pi^{-1}(\mathfrak{p})$ $\pi: A' \to A'/J$ $(\simeq A)$
$\mathfrak{p}$: ideal of A $\mathfrak{p} = \mathfrak{p}'/J$.

diagram of exact sequences

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & J \arrow[r] & \mathfrak{p}' \arrow[r] \arrow[d] & \mathfrak{p} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & J \arrow[r] & A' \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & 0 \\
 & & A'/\mathfrak{p}' \arrow[r, Rightarrow] \arrow[d] & A/\mathfrak{p} \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

We tensor this diagram with M :
vertically we have
$Tor_1^{A'}(M', A') = 0$ and ~~$Tor_1^{A}(M$~~ $Tor_1^{A}(M, A) = 0$ so,

$$\begin{tikzcd}
 & 0 \arrow[d] & & 0 \arrow[d] & \\
 & Tor_1^{A'}(M', A'/P') \arrow[r] \arrow[d] & & Tor_1^{A}(M, A/P) \arrow[d] & \\
M \otimes_A J \arrow[r] \arrow[d, "f"] & M' \otimes_{A'} P' \arrow[rr] \arrow[d, "g"] & & M \otimes_A P \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & M \otimes_A J \arrow[r] \arrow[d] & M' \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & 0 \\
 & 0 & M' \otimes_{A'} A'/P' \arrow[r, phantom, "="] & M \otimes_A A/P &
\end{tikzcd}$$

here the <u>columns</u> are exact and by hypothesis 2, the second row is exact.
Now we can apply the snake lemma in this rectangle, there is a long exact sequence

$\operatorname{Ker} f \to \operatorname{Ker} g \to \operatorname{Ker} h \to \operatorname{Coker} f \to \text{---}$

here: $\operatorname{Coker} f = 0$; $\operatorname{Ker} f = 0$

$\operatorname{Ker} g = Tor_1^{A'}(M', A'/P')$ and $\operatorname{Ker} h = Tor_1^{A}(M, A/P)$

so $Tor_1^{A'}(M', A'/P') = Tor_1^{A}(M, A/P)$.

By hypothesis <u>1</u>, this is 0.

Deformation of a subscheme

X: fixed scheme over k $\qquad$ $Y \subset X$ closed subscheme

a deformation of Y over D is a closed subscheme

$Y' \subset X \times_k D$, flat over D, such that $Y' \times_D k = Y$.

recall: If T is any scheme over k, $Y' \subset X \times T$ can be thought as a family of schemes parametrized by T, and For each ~~point~~ k-point $k \to T$, the scheme-theoretic fiber of $Y' \to T$ is

$$\begin{tikzcd} Y' \times_T k \arrow[r] \arrow[d] & k \arrow[d] \\ Y' \arrow[r] & T \end{tikzcd}$$

So a deformation of Y is a family of schemes Y' parametrized by D such that the fiber of Y' over the unique k-point of D (which is then a scheme over k) is Y, ~~and~~ flat over D.

Affine case:

X corresponds to a k-algebra B;

Y to an ideal $I \subset B$ $\qquad Y = \mathrm{Spec}(B/I)$;

$X \times_k D$ to $B' := B \otimes_k D = B[t]/t^2 = B \oplus tB$, $t^2 = 0$

Y': ideal $I' \subset B'$ such that $\qquad B = B'/tB'$

- B'/I' is flat over D
- $(B'/I') \otimes_D k = B/I$

remark: for any k-algebra C $\qquad C' = C \otimes_k D = C' \oplus tC$

and $C = C'/tC'$ then $C' \otimes_D k = C'/tC' \simeq C$.

So here: condition $(B'/I') \otimes_D k = B/I$

is equivalent to $(B'/I') \bmod t = B/I$.

$\iff$ $I' \bmod t = I$.

Since B is flat over k: by proposition 2.2, flatness is equivalent to: $0 \to B/I \xrightarrow{\times t} B'/I' \to B/I \to 0$

exact!

~~We~~ given such an I' we have a diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & I \arrow[r, "\times t"] \arrow[d] & I' \arrow[r, "\bmod t"] \arrow[d] & I \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & B \arrow[r, "\times t"] \arrow[d] & B' \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & B/I \arrow[r, "\times t"] \arrow[d] & B'/I' \arrow[r] \arrow[d] & B/I \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

remark:
$I' \bmod t = I$
$\not\Rightarrow$
$0 \to I \xrightarrow{t} I' \to I \to 0$
exact

all columns are exact (trivial)
+ two bottom rows

nine lemma: this implies that the top row is exact.

Proposition 2.3 to give $I' \subset B'$ with B'/I' flat over D and $I' \bmod t = I$ is equivalent to giving $\varphi \in \mathrm{Hom}_B(I, B/I)$. $\varphi = 0$: trivial deformation ~~i.e.~~ $I' = I \oplus tI$.

Proof: given I' as above, take $x \in I$.
Lift x to $I' \to x + ty$ for some $y \in B$.
Other liftings: $x + tz$ such that $y - z \in I$
so ~~$x+$~~ $x \mapsto y$ defines $\varphi : I \to B/I$.

Conversely given φ, let
$I' = \{ x + ky \mid x \in I, y \in B, \ y \bmod I = \varphi(x) \}$
and check that $I' \bmod k = I$ and flatness:
use the commutative diagram where this time the top row and the bottom row are exacts.
This imply that the bottom row is exact and this is the flatness condition.

Clearly $\varphi = 0 \iff I' = I \oplus kI$

<u>Non-affine case</u>

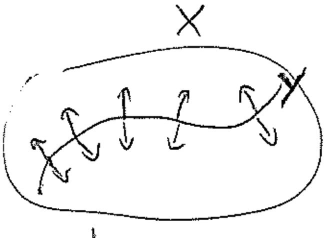
Two possibilities: glueing <u>or</u> do the same thing with ideal sheaves.

X: scheme $\quad Y$ closed subscheme $\to \mathcal{J} \subset \mathcal{O}_X$
sheaf of ideals (= "functions on X vanishing on Y").
First-order deformations are in a natural bijection
with $\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{J}, \mathcal{O}_Y)$ with $\mathcal{O}_Y = \mathcal{O}_X / \mathcal{J}$

Remark: this ~ global sections of $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{J}, \mathcal{O}_Y)$.
but this $= \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{J}/\mathcal{J}^2, \mathcal{O}_Y)$
and this is called the <u>normal sheaf</u> of Y in X.

$$\mathcal{N}_{Y/X} = \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{J}/\mathcal{J}^2, \mathcal{O}_Y).$$

$\mathcal{N}_{Y/X} = (TX|Y)/TY$



Analogy with differential (Riemannian) geometry!!
If $Y \subset X$ submanifold: first order $\simeq$ point + tangent vector
normal sheaf = vectors in X orthogonal to tangent vectors to Y

$H^0(Y, N_{Y/X})$ = set of first order deformations

but also: , $\hat{=}$ tangent

if X = projective space $\mathbb{P}^n_k$, H Hilbert scheme of Y in X, P Hilbert polynomial

and Y corresponds to a point $y \in H$, then

$H^0(Y, N_{Y/X})$ is the tangent space to H at y.

Proof: representable property of H

$$\begin{array}{ccc} \mathrm{Hom}(k, H) & \simeq & \text{closed subschemes} \subset X \times k \\ y & \Leftrightarrow & Y \end{array}$$ (proper, flat over k, Hilbert polynomial P)

$f \in \mathrm{Hom}(D, H) \simeq$ closed subschemes $Y' \subset X \times D$ (proper, flat over k, Hilbert polynomial P)

+ condition that f send k to y

$\Leftrightarrow$ condition that the fiber of Y' over k is Y.

and this is the Zariski tangent space!

Case of line bundles, vector bundles, coherent sheaves

General definition: if $\mathcal{F}$ is a coherent sheaf on a scheme X and $f: X \to Z$

$\mathcal{F}$ is flat over Z if $\forall x \in X$ $\mathcal{F}_x$ is flat over $\mathcal{O}_{f(x), Z}$.

If X is a scheme over k; a deformation of $\mathcal{F}$ over D is a coherent sheaf $\mathcal{F}'$ on $X' := X \times_k D \to D$ flat over D, with a morphism $\mathcal{F}' \to \mathcal{F}$ such that $\underbrace{\mathcal{F}' \otimes_D k}_{= \mathcal{F}' \bmod t} \to \mathcal{F}$ is an isomorphism

Two deformations $\mathcal{F}_1' \to \mathcal{F}$, $\mathcal{F}_2' \to \mathcal{F}$ are equivalent if

$$\begin{tikzcd} \mathcal{F}_1' \arrow[d, "S"'] \arrow[dr] & \\ \mathcal{F}_2' \arrow[r] & \mathcal{F} \end{tikzcd}$$

Case of a line bundle $\mathcal{L}$: deformation $\Leftrightarrow$ $\mathcal{L}'$ on $X' \times D$ such that $\mathcal{L}' \otimes \mathcal{O}_X \simeq \mathcal{L}$.

Theorem.

1. Deformations of a coherent sheaf $\mathcal{F}$ over D $\Leftrightarrow$ $\mathrm{Ext}^1_X(\mathcal{F}, \mathcal{F})$
2. Deformations of a line bundle $\mathcal{L}$ over D $\Leftrightarrow$ $H^1(X, \mathcal{O}_X)$
3. Deformations of a vector bundle $\mathcal{E}$ over D $\Leftrightarrow$ $H^1(X, \mathrm{End}(\mathcal{E}))$.

Proof

1. $0 \to \mathcal{F} \xrightarrow{\times t} \mathcal{F}' \to \mathcal{F} \to 0$

~~lemma~~ proposition 2.2 flatness $\Leftrightarrow$ this is exact

but this is split! so $\mathcal{F}' \Leftrightarrow$ extensions of $\mathcal{F}$ by $\mathcal{F}$.

2. $0 \to \mathcal{O}_X \xrightarrow{\times t} \mathcal{O}_{X'} \to \mathcal{O}_X \to 0$ is exact

replace $\mathcal{O}_X \xrightarrow{\times t} \mathcal{O}_{X'}$ by $\mathcal{O}_X \xrightarrow{\alpha} (\mathcal{O}_{X'})^*$ $(D^*, \times)$

where $\alpha(x) = 1 + tx$, group homomorphism $(k, +) \to (\cancel{k}, \times)$

this is split $\to$

$0 \to H^1(X, \mathcal{O}_X) \to H^1(X', \mathcal{O}^*_{X'}) \to \underset{\mathcal{L}}{H^1}(X, \mathcal{O}^*_X) \to 0$ is exact

3. ??

$\mathcal{E}$ is locally free. goal: $\mathrm{Ext}^1(\mathcal{E}, \mathcal{E}) = \mathrm{Ext}^1(\mathcal{O}_X, \mathrm{End}(\mathcal{E}))$

$= H^1(X, \mathrm{End}(\mathcal{E}))$??

idea: vector bundles $\overset{\text{rank } n}{\hookrightarrow} H^1(X, GL(n, \mathcal{O}_X))$

$0 \to M_n(\mathcal{O}_X) \xrightarrow{\times t} M_n(\mathcal{O}_{X'}) \to M_n(\mathcal{O}_X) \to 0$

replace by $A \mapsto \exp(tA)$: $M_n(\mathcal{O}_X) \to GL(n, \mathcal{O}_{X'})$