The T^i functors

Construction: Let A, B rings, $A \to B$ homo. and $M \in B$-mod.

Goal: Define $T^i(B/A, M)$ for $i = 0, 1, 2$.

Let $R = A[x]$, where $x = \{x_i\}$ set of variables st $B = R/I$ as A-algebra.

Thus, $0 \to I \to R \to B \to 0$

Let F be a free R-mod together with a surjection $j : F \to I \to 0$ and let $Q = \mathrm{Ker}(j)$:

$$0 \to Q \to F \xrightarrow{j} I \to 0$$

Let $F_0 = \left\{\begin{array}{l}\text{submodule of } F \text{ generated by the "Koszul} \\ \text{relations": } j(a)b - j(b)a \ , \quad a, b \in F\end{array}\right\}$

By def, $j(F_0) = 0$, so $F_0 \subseteq Q$.

We will define the cotangent complex

$$L_2 \xrightarrow{d_2} L_1 \xrightarrow{d_1} L_0 \quad \text{as follows:}$$

Take $L_2 = Q/F_0$

Rmk 0 If $x \in I$, $a \in Q$ then $x = j(x')$ for $x' \in F$

Then $xa = j(x')a \equiv \underset{\overset{\parallel}{0}}{j(a)}x'$ (mod F_0)

So L_2 is a B-mod!

Take $L_1 = F \otimes_R B = F \otimes_R R/I = F/IF$

and let $d_2 : L_2 \to L_1$ induced by $Q \hookrightarrow F$:

$j(a) \in I, \forall a \in F \Rightarrow F_0 \subseteq IF$ so

$q + \Sigma \lambda_i (j(a_i)b_i - j(b_i)a_i)$

depends only on q in F/IF

Take $L_0 = \Omega_{R/A} \otimes_R B$, where

$$\Omega_{R/A} = \langle dr \,/\, r\in R\rangle_{\substack{\text{free}\\ A\text{-mod}}} \Big/ \left\{ \begin{array}{l} d(r+r') = dr + dr' \\ d(rr') = r\,dr' + r'\,dr \\ d(a) = 0 \quad , a\in A \end{array} \right\}$$

$$\begin{aligned} d : R &\longrightarrow \Omega_{R/A} \\ r &\longmapsto dr \end{aligned}$$

Map $L_1 = F/IF$ to I/I^2 and use the exact sequence

$$I/I^2 \xrightarrow{\delta} \Omega_{R/A} \otimes_R B \longrightarrow \Omega_{B/A} \longrightarrow 0$$

$b\in I$ $\qquad \bar{b} \longmapsto db \otimes 1$

to define $L_1 \xrightarrow{d_1} L_0$.

<u>Clear</u>: $(d_1 d_2)(q + F_0) = d_1(q + IF) = 0$

$j(q)=0$

$Ij(F)\subseteq I^2$

so it is a complex of B-mod.

<u>Rmk 1</u>: $L_1 = F \otimes_R B$ free B-mod ✓

$L_0 = \Omega_{R/A} \otimes_R B$ free B-mod since

$R = A[x] \Rightarrow \Omega_{R/A}$ generated by $\{dx_i\}$

<u>Finally</u>, for $M \in B$-<u>mod</u>, we define

$$\boxed{T^i(B/A, M) = h^i(\mathrm{Hom}_B(L_\bullet, M))}$$

Explicitely: $L_2 \xrightarrow{d_2} L_1 \xrightarrow{d_1} L_0$

$(Q/F_0 \xrightarrow{d_2} F/IF \xrightarrow{d_1} \Omega_{R/A} \otimes_R B)$

induces $\mathrm{Hom}_B(L_2, M) \xleftarrow{d_2^*} \mathrm{Hom}_B(L_1, M) \xleftarrow{d_1^*} \mathrm{Hom}_B(L_0, M)$

$$\begin{array}{ccccc} & & f d_1 & \longleftarrow & f \\ g d_2 & \longleftarrow & g & & \end{array}$$

so $T^0(B/A, M) = \mathrm{Ker}\, d_1^*$

$T^1(B/A, M) = \mathrm{Ker}\, d_2^* / \mathrm{Im}\, d_1^*$

$T^2(B/A, M) = \mathrm{Hom}_B(L_2, M) / \mathrm{Im}\, d_2^*$

<u>Lemma</u>: These modules are <u>indep</u> of the choise of F and R.

From the construction:

1) $\forall\, A \to B$ and $M \in B\text{-mod}$ we have:

$L_1 = F/IF \twoheadrightarrow I/I^2$ ~~and~~ $L_1 \to L_0 \to \Omega_{B/A} \to 0$

from the exact seq

$$I/I^2 \to \Omega_{R/A} \otimes_R B \to \Omega_{B/A} \to 0$$

Taking $\mathrm{Hom}(-, M)$ we obtain

$$\mathrm{Hom}_B(L_1, M) \xleftarrow{d_1^*} \mathrm{Hom}_B(L_0, M) \leftarrow \mathrm{Hom}_B(\Omega_{B/A}, M) \leftarrow 0$$

exact

$$\Rightarrow \ker d_1^* = \boxed{T^0(B/A, M) = \mathrm{Hom}_B(\Omega_{B/A}, M)}$$

$$\overset{\shortparallel}{\mathrm{Der}_A(B, M)}$$

In particular, $T^0(B/A, B) = \mathrm{Hom}_B(\Omega_{B/A}, B)$

$$\overset{\shortparallel}{T_{B/A}}$$

Tangent module of B over A $\rightsquigarrow T_{B/A}$

2) If B is a polynomial ring over A, we can take $R = B \Rightarrow I = 0$, $F = 0$, so $L_2 = L_1 = 0$

$\Rightarrow T^i = 0$ for $i = 1, 2$ and any M.

3) If $f: A \to B$ is surjective and $I = \mathrm{Ker}(f)$ then $T^0(B/A, M) = 0 \;\forall M$ and $T^1(B/A, M) = \mathrm{Hom}_B(I/I^2, M)$.

In fact, we can take $R = A \Rightarrow L_0 = \underbrace{\Omega_{A/A}}_{=0} \otimes_A B = 0 \Rightarrow T^0 = 0$

*) $0 \to Q \to F \to I \to 0$ gives $Q \otimes_A B \to F \otimes_A B \to \underset{\overset{\shortparallel}{I/I^2}}{I \otimes_A B} \to 0$

*) $Q \twoheadrightarrow Q/F_0 \Rightarrow Q \otimes_A B \twoheadrightarrow Q/F_0 \otimes_A B = Q/F_0$

So we obtain an exact sequence

$$L_2 \to L_1 \to I/I^2 \to 0$$

$$\Rightarrow \mathrm{Hom}_B(L_2, M) \xleftarrow{d_2^*} \mathrm{Hom}_B(L_1, M) \leftarrow \mathrm{Hom}_B(I/I^2, M) \leftarrow 0$$

$$\Rightarrow \ker d_2^* / \underset{\overset{\shortparallel}{0}}{\boxed{\mathrm{Im}\, d_1^*}} = \mathrm{Hom}_B(I/I^2, M).$$

In particular, $T^1(B/A, B) = \mathrm{Hom}_B(I/I^2, B) := N_{B/A}$ normal module of Spec B in Spec A

4) If A local ring, $B = A/I$, where $I = (a_1, \to a_r)$ gen. by $a_1, \to a_r$ regular sequence $\Rightarrow T^2(B/A, M) = 0 \ \forall M$

In this case $Q = F_0$, so $L_2 = 0$ and hence $T^2 = 0$.

5) By construction $T^i(B/A, -) : B\text{-}\underline{\text{mod}} \to B\text{-}\underline{\text{mod}}$ is a covariant additive functor. If $0 \to M' \to M \to M'' \to 0$ is exact, then

$$0 \to \mathrm{Hom}(L_0, M') \to \mathrm{Hom}(L_0, M) \to \mathrm{Hom}(L_0, M'') \to 0$$

$$0 \to \mathrm{Hom}(L_1, M') \to \mathrm{Hom}(L_1, M) \to \mathrm{Hom}(L_1, M'') \to 0$$

$$0 \to \mathrm{Hom}(L_2, M') \to \mathrm{Hom}(L_2, M) \to \mathrm{Hom}(L_2, M'')$$

are exact by Remark 1. So we obtain a sequence

$$0 \to \mathrm{Hom}_B(L_\bullet, M') \to \mathrm{Hom}_B(L_\bullet, M) \to \mathrm{Hom}_B(L_\bullet, M'') \to 0$$

~~exact~~ of complexes, exact except for $\mathrm{Hom}(L_2, M) \to \mathrm{Hom}(L_2, M'')$ and thus we obtain a long exact sequence:

$$0 \to T^0(B/A, M') \to T^0(B/A, M) \to T^0(B/A, M'') \to$$
$$\to T^1(B/A, M') \to T^1(B/A, M) \to T^1(B/A, M'') \to$$
$$\to T^2(B/A, M') \to T^2(B/A, M) \to T^2(B/A, M'').$$

<u>Proof of Lemma</u>: T^i is indep of the choice

i) <u>of F</u> (keeping R fixed):

Let $F \xrightarrow{j} I$ and $F' \xrightarrow{j'} I$ be two choices

Comparing F and F' $\Leftrightarrow$ Comparing F and $F \oplus F'$

F' free $\Rightarrow \exists\, \varphi : F' \to F$ st $j' = j\varphi$. So by changing the base of $F \oplus F'$ by replacing each generator e' of F' by $e' - \varphi(e')$, we may assume that $(j, j') = (j, 0)$

So we have

$$\begin{tikzcd}
0 \arrow[r] & Q \oplus F' \arrow[r] \arrow[d] & F \oplus F' \arrow[r, "{(j,0)}"] \arrow[d, "pr_1"] & I \arrow[r] \arrow[d, "id"] & 0 \\
0 \arrow[r] & Q \arrow[r] & F \arrow[r, "j"] & I \arrow[r] & 0
\end{tikzcd}$$

$\Rightarrow \mathrm{Ker}(j, 0) = Q \oplus F'$ and $(F \oplus F')_0 = F_0 + IF'$ ↖

$(j,0)(f_1, f_1') \cdot (f_2, f_2') - (j,0)(f_2, f_2') \cdot (f_1, f_1')$

Let $L_\bullet'$ the new complex

$$\Rightarrow L_2' = (Q \oplus F')/(F_0 + IF') = L_2 \oplus (F'/IF')$$

$$L_1' = (F \oplus F') \otimes_R B = (F \otimes_R B) \oplus (F' \otimes_R B) = L_1 \oplus (F' \otimes_R B)$$

$$L_0' = L_0$$

Since $F' \otimes_R B = F'/IF'$ is free B-mod

$L_\bullet'$ is obtained from $L_\bullet$ by taking direct sum with the free acyclic complex $0 \to F' \otimes_R B \to F' \otimes_R B$

So by taking Hom and h^i we obtain the same.

ii) of R: As before, compare $R = A[x]$ and $R' = A[y]$

$\Leftrightarrow$ Compare $R = A[x]$ with $R'' = A[x,y]$

After change of variables we can suppose that all the y_i goes to 0 by $A[x,y] \twoheadrightarrow B$.

So $\ker(R'' \to B)$ is gen. by I and all the y_i's.

By i) we can choose $F \twoheadrightarrow I$ as we want.

Take F to be any free R-mod st $F \twoheadrightarrow I$ and F' a free R''-mod on the same number of generators as F, and G' a free R''-mod on the index set of ~~variables~~ the y_i variables. Then

$$\begin{tikzcd}
0 \arrow[r] & Q' \arrow[r] & F' \oplus G' \arrow[r] & IR'' + yR'' \arrow[r] & 0 \\
0 \arrow[r] & Q \arrow[r] \arrow[u] & F \arrow[r] \arrow[u] & I \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

By similar computations:

$$L_2 = L_2', \quad L_1' = L_1 \oplus \underbrace{(G' \otimes_{R''} B)}_{\text{free}}, \quad L_0' = L_0 \oplus \underbrace{(\Omega_{A[y]/A} \otimes B)}_{\text{free}}$$

We conclude as in i) that we obtain isomorphic T^i ■

Theorem: If $A \to B \to C$ are rings, and $M \in C$-mod, there is an exact sequence of C-modules:

$$0 \to T^0(C/B, M) \to T^0(C/A, M) \to T^0(B/A, M)$$
$$\to T^1(C/B, M) \to T^1(C/A, M) \to T^1(B/A, M)$$
$$\to T^2(C/B, M) \to T^2(C/A, M) \to T^2(B/A, M).$$

Proof: Choose: i) $A[x] \to B \to 0$ st $B = A[x]/I$
(Sketch) and $0 \to Q \to F \to I \to 0$

ii) $B[y] \to C \to 0$ st $C = B[y]/J$

and $0 \to P \to G \to J \to 0$

Then, $A[x,y] \to B[y] \to C$ gives a surjection st $C = A[x,y]/K$

$\Rightarrow 0 \to I[y] \to K \to J \to 0$ exact

Take F', G' to be free $A[x,y]$-mod on the same index sets as F and G, resp.

For $G \twoheadrightarrow J$ choose a lifting $G' \to K$ and consider
$\hat{}$ $B[y]$ free mod

the natural map $F' \to K \Rightarrow F' \oplus G' \to K$ gives a surjectio

Let S be its kernel: $0 \to S \to F' \oplus G' \to K \to 0$.

Then, we have induced maps of complexes

$$L_\bullet(B/A) \otimes_B C \to L_\bullet(C/A) \to L_\bullet(C/B)$$

•) On the deg 0 level:

$$\Omega_{A[x]/A} \otimes_B C \to \Omega_{A[x,y]/A} \otimes_B C \to \Omega_{B[y]/B} \otimes_B C$$

gen. by $\{dx_i\}$ gen. by $\{dx_i, dy_i\}$ gen. by $\{dy_i\}$

So is split exact.

•) On the deg 1 level:

$$F \otimes C \to (F' \oplus G') \otimes C \to G \otimes C$$

is split exact by construction.

•) On the deg 2 level

$$(Q/F_0) \otimes_B C \to S/(F' \oplus G')_0 \to P/G_0$$

we have surjectivity in the right map, because $S \to P$ is surjective.

Moreover, by standard computations, the middle is exact

Taking $\mathrm{Hom}(-, M)$ and cohomology we obtain the desired long exact sequence. ■

<u>Corollary</u>: Suppose $A = k[x_1, \to x_n]$, $B = A/I$

Then $\forall M$, the sequence

$$0 \to T^0(B/k, M) \to \mathrm{Hom}(\Omega_{A/k}, M) \to \mathrm{Hom}(I/I^2, M) \to T^1(B/k, M) \to 0$$

is exact and

$$T^2(B/A, M) \xrightarrow{\sim} T^2(B/k, M)$$

<u>Proof</u>: $k \to A \to B$ induces the long exact sequence

$$0 \to T^0(\overset{0\ \text{by ③}}{\cancel{B/A}}, M) \to T^0(B/k, M) \to \underset{\text{by ①} \to \ \mathrm{Hom}(\Omega_{A/k}, M)}{T^0(A/k, M)} \to$$

$$\to \underset{\mathrm{Hom}(I/I^2, M)\ \leftarrow \text{by ③}}{T^1(B/A, M)} \to T^1(B/k, M) \to T^1(\overset{0\ \text{by ②}}{\cancel{A/k}}, M) \to$$

$$\to T^2(B/A, M) \to T^2(B/k, M) \to T^2(\overset{0\ \text{by ②}}{\cancel{A/k}}, M) \quad \blacksquare$$

<u>Rmk</u>: If A noeth, B fin. gen A-alg and M fin-gen. B-mod $\Rightarrow T^i(B/A, M)$ is a fin. gen. B-mod.

<u>Notation</u>: $T^i(B/A, B) := T^i_{B/A}$

$$T^i(B/k, B) = T^i_{B/k} := T^i_B$$

$$T^0_{B/A} := T_{B/A}$$

<u>Rmk</u>: If $f: X \to Y$ is a projective morphism, ie, $X \subseteq \mathbb{P}^n_Y$ for some n.

We can make the same construction, by using $\mathcal{I}_X \subseteq \mathcal{O}_{\mathbb{P}^n_Y}$ sheaf of ideals, to define a complex $\mathcal{L}_\bullet$ (of sheaves) and define

$\mathcal{T}^i(X/Y, \mathcal{F}) = h^i(\mathcal{H}om_X(\mathcal{L}_\bullet, \mathcal{F}))$ for $\mathcal{F} \in \mathcal{O}_X$-<u>mod</u>.