The infinitesimal lifting property

Louis Clément Lefèvre

Artin rings

A is an Artin ring $\Leftrightarrow$ every decreasing sequence of ideals is stationary.

Let k: field of characteristic 0.

Def: A is an Artin local k-algebra if A is a k-algebra, with a unique maximal m, finite dimensional over k and $A/m \simeq k$.

Basic examples: k; $D := k[t]/t^2$; $A_n := k[t]/t^{n+1}$.

There is a structure map $k \to A$ and a projection $A \to A/m \simeq k$, we call it reduction mod m.

There are natural maps $A_n \to A_r$ if $n \geq r$; this is reduction mod t^{r+1}.

Proposition: the maximal ideal m is nilpotent (there exists n such that $m^n = 0$)

Proof: the (m^i), $i \geq 0$ form a decreasing sequence of ideals, and when it becomes stationary, $m^n = 0$. $\square$

So there is a sequence of projections

$$A = A/m^n \to A/m^{n-1} \to \cdots \to A/m^2 \to A/m \simeq k.$$

and at each step the kernel I satisfies $I^2 = 0$

Given elements $a_1, \ldots, a_n \in m$ there is a unique well-defined map $\varphi : k[[x_1, \ldots, x_n]] \to A$ such that $\varphi(x_i) = a_i$.

Deformations and thickenings

Let A be a local ~~k-algebra~~ Artin k-algebra.
$\mathrm{Spec}(A)$ has a unique point which is closed, it is a scheme over k with a map $\mathrm{Spec}(k) \to \mathrm{Spec}(A)$ dual to the reduction mod m; $\mathrm{Spec}(A)$ is called a punctual scheme.

Def: If X is a scheme over k: a <u>deformation of X over A</u> is a scheme X', flat over A with a map $i: X \to X'$ such that $i \times_A k : X \to X' \times_A k$ is an isomorphism. i is then a closed immersion, so we can think of $X \subset X'$ as closed subscheme and the central fiber of X' over the unique point of $\mathrm{Spec}(A)$ is X.
Also a deformation is a pull-back diagram

$$\begin{array}{ccc} X & \xrightarrow{i} & X' \\ \downarrow & & \downarrow \text{flat} \\ k & \to & A \end{array}$$

In this form it generalises very well: if S is any base scheme, a flat map $X' \to S$ is a family of schemes parametrised by S. If $\Delta : \mathrm{Spec}(k) \to S$ is a point, ~~asking~~ a pull-back diagram

$$\begin{array}{ccc} X & \to & X' \\ \downarrow & & \downarrow \text{flat} \\ k & \xrightarrow{\Delta} & S \end{array}$$

is asking that the fiber over $\Delta \in S$ to be X. In a differential-geometric setting, ~~a defo~~ a deformation over a base S is expressed by exactly the same diagram where X, X', S are manifolds and you replace $k \to S$ by a map $pt \to S$; then $X' \times_S k$ is the fiber over pt.

Two deformations X'_1, X'_2 are <u>equivalent</u> if there is an isomorphism $f: X'_1 \to X'_2$ over A such that $i_2 = f \circ i_1$.

A deformation is <u>trivial</u> if it is isomorphic to the trivial deformation, which is $X \times_k A$.

<u>Def.</u> If Y is any scheme, an <u>infinitesimal thickening</u> is a scheme Y' such that Y is a closed subscheme of Y' and the ideal sheaf $\mathcal{I}_{Y/Y'}$ is nilpotent.

Basic examples: take for Y' a deformation of Y over A. $\mathrm{Spec}(A)$ is a thickening of $\mathrm{Spec}(k)$.

<u>Def.</u> If $f: Y \to X$ is a morphism of schemes and $Y \subset Y'$ is an infinitesimal thickening we say that f lifts to Y' if there exists $g: Y' \to X$ such that $g|_Y = f$.

Our <u>goal</u> is to show that (k algebraically closed)

1. If X is nonsingular over k, affine of finite type, Y is affine over k and $Y \subset Y'$ is an infinitesimal thickening and $f: Y \to X$ then f lifts to Y'.
2. If X is a scheme of finite type over k such that for any $f: Y \to X$ with $Y = \mathrm{Spec}(A)$ (A: Artin local k-algebra) and every infinitesimal thickening $Y \subset Y'$, f lifts to Y'; then X is nonsingular ~~(in fact if f sends the unique k-point y of Y to x~~

(in fact if we reduce to those f which send the unique k-point y of Y to ~~see~~ a fixed $x \in X$ the the conclusion is that X is nonsingular at x).

<u>Corollary</u>: if X is affine, nonsingular over k, then every deformation of X is trivial.

<u>Lemma.</u> (Hartshorne, exercise 4.2) if X_1, X_2 are schemes of finite type over k, flat over A, and $f: X_1 \to X_2$ is an A-morphism ~~such that~~ that induces an isomorphism $f \underset{A}{\otimes} k : X_1 \underset{A}{\times} k \to X_2 \underset{A}{\times} k$ then f is an isomorphism.

<u>Proof:</u>

<u>Proof of corollary:</u> if X' is a deformation over A, it is a thickening. Lift $id: X \to X$ to $g: X' \to X$.
Then g induces $g': X' \to X \underset{k}{\times} A$
such that $g' \underset{A}{\times} k : X \to X$ is id.
So by the lemma, $X' \simeq X \underset{k}{\times} A$.

<u>Recall on smoothness</u>
k algebraically closed; X: scheme of finite type $/k$.

- X is nonsingular at $P \iff \mathcal{O}_{X,P}$ is a regular local ring
  (A, m) regular local ring $\iff \dim_{Krull} A = \dim_{A/m}(m/m^2)$
  (and this: minimal number of generators of m, by Nakayama's lemma)

$\Leftrightarrow$ Jacobian criterion ...

- X is non singular $\Leftrightarrow$ $\Omega^1_{X/k}$ is locally free of rank n at every point of X. $(n = \dim X)$

- If $Y \subset X$ irreducible closed subscheme of X nonsingular, with a sheaf of ideals $\mathcal{I}$, then $\mathcal{I}$ is non singular $\Leftrightarrow$
  i. $\Omega^1_{Y/k}$ is locally free and
  ii. $0 \to \mathcal{I}/\mathcal{I}^2 \to \Omega^1_{X/k} \otimes \mathcal{O}_Y \to \Omega_{Y/k} \to 0$
  is exact on the left.

Remark 1: it is always right-exact. and the T^i functors behave like left derived functors.

Remark 2: when proving $\Leftarrow$, condition ii is used to show that the rank is constant.

Proof of the first part

- step 1: reduce to the affine case

  $X = \mathrm{Spec}(A)$, $Y = \mathrm{Spec}(B)$, $Y' = \mathrm{Spec}(B')$,

  $B = B'/I$, I nilpotent $I^n = 0$.

  We give $f: A \to B$ and we want to ~~fit~~ lift it to $g: A \to B'$ such that

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[dr, "f"'] & B' \arrow[d, "\mathrm{mod}\, I"] \\ & B \end{tikzcd}$$

- step 2: lift "step by step": there is a sequence of surjective maps

$$B' = B/I^n \to B'/I^{n-1} \to \cdots \to B'/I^2 \to B'/I = B$$

and we lift in the reverse order;
so we can reduce to the case $I^2 = 0$

- case $A = k[x_1, x_n]$ (A: affine space) easy!!
because $f: A \to B$ is determined by the $b_i := f(x_i)$.
so choose $b'_i \in B'$ that lift b_i and take
$g(x_i) = b'_i$, this defines $g: A \to B'$.

- general case: write $A = P/J$, $P = k[x_1, x_n]$,
J an ideal (this $\Leftrightarrow$ closed embedding of A in the affine space $\mathrm{Spec}(P)$)
$f: A \to B$ gives a map $P \to B$ which lifts to $h: P \to B'$,
and h induces a map $J \to I$, so that

$$\begin{array}{ccccccccc} 0 & \to & J & \to & P & \to & A & \to & 0 \\ & & & & \downarrow h & & \downarrow f & & \\ 0 & \to & I & \to & B' & \to & B & \to & 0 \end{array} \qquad \begin{array}{l} h(J) \subset I \\ h(J^2) \subset I^2 \end{array}$$

so defines $\bar{h}: J/J^2 \to I$. We now have the exact sequence
$$0 \to J/J^2 \to \Omega^1_{P/k} \otimes_P A \to \Omega^1_{A/k} \to 0.$$
B' is a P-module via h, B is an A-module via f; we dualize
$$0 \to \mathrm{Hom}_A(\Omega^1_{A/k}, I) \to \mathrm{Hom}_P(\Omega^1_{P/k}, I) \to \mathrm{Hom}_A(J/J^2, I) \to 0.$$
$$\bar{h} \in$$
this sequence is exact, because they are projective A- (locally free) A-modules.
So we can take $\theta \in \mathrm{Hom}_P(\Omega^1_{P/k}, I)$ that projects onto $\bar{h}$.
(θ is a derivation from P into I).
Define $h': P \to B'$ by $h' = h - \theta$.

then $h'(J) = 0$ and we can check that h' is a ring homomorphism which induces $\overline{h'} : P/J \to B'$ which is the desired lifting. 4

<u>Proof of the second ~~case~~ part</u>

- first step: reduce to affine (and local) case:
  (A, m) is a local k-algebra (finite type / k, residue field k)
  $Y = \mathrm{Spec}(B)$ B' = local Artin k-algebra
  $Y' = \mathrm{Spec}(B')$ thickening, $B = B'/I$.
  and we can reduce to $I^2 = 0$.
  Assume that every $A \to B$ lifts to $A \to B' \to B$.
  We want to show that A is a regular local ring

<u>Recall:</u> if $a_1, \ldots, a_n$ is a minimal set of generators of m then A is regular $\iff$ $\underset{(\text{completion}/m)}{\widehat{A}} \simeq k[[x_1, \ldots, x_n]]$

(and the isomorphism sends x_i to a_i).

This is analogue to an inverse function theorem.
See: Cohen structure theorem.

Let $a_1, \ldots, a_n$ be such a sequence, it induces naturally a surjective homomorphism $f : P = k[[x_1, \ldots, x_n]] \to \widehat{A}$ with $f(x_i) = a_i$.

If $n = (x_1, \ldots, x_n)$ is the maximal ideal of P, then

f induces an isomorphism $P/n^2 \xrightarrow{\sim} \hat{A}/m^2 \simeq A/m^2$

(as k-vector spaces the first one is spanned by k and $x_1, \ldots, x_n$; the second one by k and $a_1, \ldots, a_n$).

Now we use the hypothesis to lift $A \to A/m^2 \simeq P/n^2$ successively to $A \to P/n^i$

$$
\begin{tikzcd}
A \arrow[d] \arrow[r] \arrow[rr, dashed] & P/n^2 \arrow[dl] & P/n^3 \arrow[l] & \cdots \arrow[l] \\
P/n \simeq k & & &
\end{tikzcd}
$$

and this induces a map $A \longrightarrow \varprojlim P/n^i = P$

which extends to $g : \hat{A} \to P$.

So we have $P \xrightarrow{f} \hat{A} \xrightarrow{g} P$ ~~so~~ and

$g \circ f : P \to P$ induces an automorphism of P/n^2

this implies (by the next lemma) that $g \circ f$ is an automorphism of P, so f is injective and $\hat{A} \simeq k[[x_1, \ldots, x_n]]$ $\square$

<u>Lemma</u>: if $h : P \to P$ induces an ~~isomorphism~~ automorphism $\bar{h} : P/n^2 \to P/n^2$ then h itself is an automorphism

<u>Proof</u>, by hand case ~~$n=1$~~ $P = k[[x]]$. h is determined by $F = h(x)$. As vector space $P/n^2 = \mathrm{Vect}(k1, x)$ and ~~this~~ if $F = a_0 + a_1 x + \ldots$ then $\bar{h}$ is the linear map

$\lambda + \mu x \mapsto (\lambda + \mu a_0) + \mu a_1 x$ which is invertible iff $a_1 \neq 0$.

In case $a_1 \neq 0$ we can write $x = \frac{1}{a_1}(F - a_0)$ and then define an inverse to h by sending x to $\frac{1}{a_1}(F - a_0)$.

Case $P = k[[x_1, \ldots, x_n]]$ is similar by writing matrices.