Deformation of Rings

<u>Recalls</u>: $A \to B$ homo. of rings and $M \in B\text{-}\underline{\text{mod}}$

Let $R = A[X]$, $X = \{x_i\}_{i\in I}$ st $B = R/I$ and F free R-mod st $0 \to Q \to F \xrightarrow{j} I \to 0$ is exact. Let

$$F_0 = \langle j(a)\,b - j(b)\,a \rangle_{a,b\in F} \subseteq Q_0$$

and consider the cotangent complex

$$\begin{array}{ccccc} L_2 & \xrightarrow{d_2} & L_1 & \xrightarrow{d_1} & L_0 \\ \| & & \| & & \| \\ Q/F_0 & & F\otimes_R B = F/IF & & \Omega_{R/A}\otimes_R B \end{array}$$

$L_\bullet$ complex $\Rightarrow \mathrm{Hom}_B(L_\bullet, M)$ is a complex. We define

$$\boxed{T^i(B/A, M) = h^i(\mathrm{Hom}_B(L_\bullet, M))}$$

They satisfy several properties, <u>like</u>: If $A = k[x_1, \to x_n]$, $B = A/I$ then $\forall M \in B\text{-}\underline{\text{mod}}$ there is an exact sequence

$$0 \to T^0(B/k, M) \to \mathrm{Hom}(\Omega_{A/k}, M) \to \mathrm{Hom}(I/I^2, M) \to T^1(B/k, M) \to 0$$

and $T^2(B/A, M) \xrightarrow{\sim} T^2(B/k, M)$.

<u>Thm</u>: $(k = \overline{k})$ Let $X = \mathrm{Spec}\, B$ affine scheme $/k$. Then, X is nonsingular $\iff T^1(B/k, M) = 0 \quad \forall M \in B\text{-}\underline{\text{mod}}$

Moreover, X nonsingular $\Rightarrow T^2(B/k; M) = 0 \quad \forall M$.

<u>Proof</u>: Write $B = A/I$, where $A = k[x_1, \to x_n]$

X nonsingular $\iff 0 \to I/I^2 \xrightarrow{f} \Omega_{A/k}\otimes_A B \to \Omega_{B/k} \to 0$ exact and $\Omega_{B/k}$ locally free. ($\Leftrightarrow$ projective module)

$\iff 0 \to I/I^2 \to \Omega_{A/k}\otimes_A B \to \Omega_{B/k} \to 0$ is split exact.
$\left(\begin{smallmatrix}\Omega_{A/k} \\ \text{free } A\text{-mod}\end{smallmatrix}\right)$

In other hand,

$T^1(B/k, M) = 0 \ \forall M \in B\text{-}\underline{\text{mod}} \iff \mathrm{Hom}(\Omega_{A/k}, M) \twoheadrightarrow \mathrm{Hom}(I/I^2, M)$ surjective $\forall M \in B\text{-}\underline{\text{mod}}$.

So X nonsingular $\Rightarrow T^1(B/k, M) = 0 \ \forall M$

Now, $T^1(B/k, M) = 0$ implies ~~Hom($\Omega_{A/k}$, I/I^2)~~

$\mathrm{Hom}(\Omega_{A/k}, I/I^2) \to \mathrm{Hom}(I/I^2, I/I^2)$ surjective

$\Rightarrow \exists\, g \in \mathrm{Hom}(\Omega_{A/k}, I/I^2)$ st $fg = id_{I/I^2}$

$\Rightarrow$ The sequence is split ~~exact~~

Now, if X nonsingular, $\forall x \in X$, I_x is gen by $n - r = \dim A - \dim B$ elements in the regular local ring $A_x \Rightarrow T^2(B_x/A_x, M) = 0 \;\forall B_x$-mod M

$\Rightarrow T^2(B/A, M) = 0 \;\forall B$-mod M

$\quad \wr\!\wr$

$T^2(B/k, M)$ ∎

Coro: B local k-algebra, $B/m \simeq k$ $(k = \bar{k})$

Then, B regular local ring $\Leftrightarrow T^1(B/k, M) = 0 \;\forall M \in B$-mod

And, in this case $T^2(B/k, M) = 0 \;\forall M$.

Def: A morphism of noeth. schemes $f: X \to Y$ is smooth if

•) f is of finite-type

•) f is flat

•) $\forall y \in Y$, the geometric fiber $X_y \otimes_{k(y)} \overline{k(y)}$ is nonsingular over $\overline{k(y)}$.

Thm: A morphism of finite type $f: X \to Y$ of noeth. schemes is smooth $\Leftrightarrow$ f is flat and $T^1(X/Y, \mathcal{F}) = 0$ $\forall \mathcal{F} \in \underline{\mathrm{Coh}}(X)$. In this case, then also $T^2(X/Y, \mathcal{F}) = 0$ $\forall \mathcal{F} \in \underline{\mathrm{Coh}}(X)$.

Proof: The question is local, so we can assume $X = \mathrm{Spec}\, B$, $Y = \mathrm{Spec}\, A$ and $f: A \to B$

$(\Rightarrow)$ Base change + Lemma of Dévissage

$(\Leftarrow)$ Base change + Thm above. ∎

Def: If A regular local ring and $B = A/I$ we say that B is a local complete intersection (l.c.i) in A if I can be gen. by $\dim A - \dim B$ elements.

Thm: Let $(A, \mathfrak{m})$ be a regular local k-alg with $A/\mathfrak{m} = k = \bar{k}$ and $B = A/I$. Then, B is a l.c.i. in A $\Longleftrightarrow T^2(B/k, M) = 0 \ \forall M \in B$-mod.

Proof: A regular $\Rightarrow T^i(A/k, M) = 0$ for $i = 1, 2$ $\forall M$

Moreover, $k \to A \to B$ induces a long exact sequence

$$\dots \to \underbrace{T^1(A/k, M)}_{0} \to T^2(B/A, M) \to T^2(B/k, M) \to \underbrace{T^2(A/k, M)}_{0}.$$

$(\Rightarrow)$ B l.c.i in $A \Rightarrow T^2(B/A, M) = 0 \ \forall M$

$(\Leftarrow)$ $T^2(B/k, M) = 0 \Leftrightarrow T^2(B/A, M) = 0 \ \forall M$

To compute T^2 we can choose: $R = A$, $I = I$, $F \xrightarrow{j} I$ a map to a minimal set of generators $(a_1, \dots, a_s)$ of I. $\Rightarrow \ker(j) = Q \subseteq \mathfrak{m}F$.

$F \simeq A^s \xrightarrow{j} I$
$f_i \mapsto a_i$
$x = \sum \lambda_i f_i \in Q$
$\Rightarrow j(x) = \sum \lambda_i e_i = 0 \Rightarrow \lambda_i \notin A^*$

Now, $T^2(B/A, M) = \operatorname{Hom}(Q/F_0, M) / d_2^*(\operatorname{Hom}(F/IF, M)) = 0 \ \forall M$ implies that $\operatorname{Hom}(F/IF, M) \xrightarrow{d_2^*} \operatorname{Hom}(Q/F_0, M)$ is surjective $\forall M$. In part. for $M = Q/F_0$ we obtain that $\exists\, p : F/IF \to Q/F_0$ st $p \circ d_2 = id_{Q/F_0}$.

As $Q \subseteq \mathfrak{m}F$ we have

$$\begin{tikzcd}
Q/F_0 \arrow[r, "d_2"] \arrow[dr, "id_{Q/F_0}"'] & \mathfrak{m}(F/IF) \arrow[d, "p"] \\
& \mathfrak{m}(Q/F_0)
\end{tikzcd}$$

Nakayama $\Rightarrow Q/F_0 = 0$

Koszul complex: $F \wedge F \xrightarrow{\varphi} F \xrightarrow{j} A$

$$x \wedge y \overset{\varphi}{\longmapsto} j(y)x - f(x)y$$

$F_0 = \operatorname{Im}(\varphi)$, $Q = \ker(j)$

$\Rightarrow H_1(\underline{a}, A) = Q/F_0 = 0 \Rightarrow \underline{a} = (a_1, \dots, a_s)$ is regular ∎

Rmk: $f: X \to Y$ is a relative l.c.i. morphism if it is flat and the geometric fibers are l.c.i. schemes.
As before, $f: X \to Y$ is a relative l.c.i morphism $\iff T^2(X/Y, \mathcal{F}) = 0 \quad \forall \mathcal{F} \in \underline{Coh}(X)$.

Recall: If $X \in \underline{Sch}_k$, a deformation of X over A is a scheme X' flat over A with $i: X \hookrightarrow X'$ closed immersion st
$$i \otimes_A k : X \to X' \otimes_A k \text{ is an isom.}$$

$$(X_1', i_1) \sim (X_2', i_2) \iff \exists f: X_1' \to X_2' \text{ isom. st } i_2 = f \circ i_1.$$
$$\begin{tikzcd} X_1' \arrow[rr, "f"] \arrow[dr] & & X_2' \arrow[dl] \\ & A & \end{tikzcd}$$

Ex: $A = D = k[t]/t^2$, $X = \operatorname{Spec} B \longrightarrow X' = \operatorname{Spec} B'$ st
i) $D \to B'$ flat
ii) $B' \to B$ homo. st $B' \otimes_D k \xrightarrow{\sim} B$

Seen: (i) $\iff 0 \to B \xrightarrow{\cdot t} B' \xrightarrow{\pi} B \to 0$ exact (*)
$$\begin{tikzcd} B \arrow[d, phantom, "\wr"] & B' \arrow[d, phantom, "\wr"] & B \arrow[d, phantom, "\wr"] \\ \text{ideal st } B^2=0 & \text{D-alg, k-alg} & \text{k-alg} \end{tikzcd}$$

We can recover the D-alg structure of B' in a unique way compatible with the exact seq. (*): just specify $\cdot t$ in B' by $B' \xrightarrow{\pi} B \xrightarrow{\cdot t} B'$ ✓

So, {Deformation of B over D}/$\sim$ $\longleftrightarrow$ {exact sequences (*)}/$\sim$

Def: Let $A \in \underline{Ring}$, $B \in \underline{A\text{-alg}}$, $M \in \underline{B\text{-mod}}$. An extension of B by M as A-alg. is an exact seq.
$$0 \to M \to B' \to B \to 0$$
where $B' \to B$ homo. of A-alg, and $M \subseteq B'$ ideal st $M^2 = 0$.
$B' \sim B'' \iff \exists\, B' \xrightarrow{\simeq} B''$ isom. compatible with the id_B, id_M in the exact sequence.

The trivial extension is $B' = B \oplus M$ with ring structure
$$(b, m) \cdot (b_1, m_1) = (bb_1, bm_1 + b_1 m)$$

<u>Thm</u>: Equivalence classes of extensions of B by M as A-algebras are in natural ~~corr~~ 1-1 corresp. with the elements in $T^1(B/A, M)$. The trivial extension corresp to $0 \in T^1(B/A, M)$.

<u>Proof</u>: Let $A[x] \to B$ be a surjection, let $\{e_i\}_{i\in I} \subseteq M$ generators of M as B-mod, and let $y = \{y_i\}_{i\in I}$ be a set of indeterminates.

<u>Note that</u>: If B' is any extension of B by M $\Rightarrow \exists f : A[x,y] \to B'$ surjective (<u>not unique</u>) st

$$\begin{tikzcd}
0 \arrow[r] & (y) \arrow[r] \arrow[d, "y_i \mapsto e_i"'] & A[x,y] \arrow[r] \arrow[d, "f"] & A[x] \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M \arrow[r] \arrow[d] & B' \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

is commutative.

<u>Step 1</u>: Classify the quotients $f : A[x,y] \to B'$

<u>Step 2</u>: Given B', how many ways are there to express B' as a quotient of B'?

<u>Step 1</u>: We complete the diagram with the kernels

$$\begin{tikzcd}
0 \arrow[r] & Q \arrow[r] \arrow[d] & I' \arrow[r] \arrow[d, "f"] & I \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & (y) \arrow[r] \arrow[d] & A[x,y] \arrow[r] \arrow[d] & A[x] \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M \arrow[r] & B' \arrow[r] & B \arrow[r] & 0
\end{tikzcd}$$

Quotients $f : A[x,y] \to B' \longleftrightarrow$ ideals $I' \subseteq A[x,y]$

Moreover we have an splitting $A[x] \to A[x,y]$

Then, $I' \subseteq A[x,y] \longleftrightarrow \varphi \in \mathrm{Hom}_{A[x]}(I,M)$

$I' \longmapsto \underline{\varphi_{I'}}$: Take $p \in I$ and $p' \in I'$ a lift

$\Rightarrow p' = p + \sum \lambda_i y_i = p + \sum \mu_i y_i$, $\lambda_i \in A[x,y]$

$\Rightarrow \lambda_i - \mu_i \in Q$

$p \longmapsto p + \sum \lambda_i y_i \longmapsto \sum \lambda_i y_i$, $\quad p + \sum \lambda_i y_i \in A[x] \oplus (y)$

well def in $(y)/Q = M$

$I'_\varphi \longleftrightarrow \varphi$

$$I'_\varphi = \left\{ p + \sum \lambda_i y_i \;\middle/\; \begin{array}{l} p \in I,\ \lambda_i \in A[x,y] \\ \text{st } \sum \lambda_i y_i = \varphi(p) \\ \text{in } (y)/Q \end{array} \right\}$$

and $\mathrm{Hom}_{A[x]}(I,M) = \mathrm{Hom}_B(I/I^2, M)$ as $M^2 = 0$

<u>Step 2</u>: <u>Seen</u>. If $B' \to B$ surjective, $A[x] \to B$ homo, then if $f, h : A[x] \to B'$ are two liftings of $A[x] \to B$ $\Rightarrow$ $\theta = h - f$ is a A-derivation of $A[x]$ to $\ker(B' \to B) = M$

<u>Now</u>, $A[x] \to A[x,y]$ implies that any $f : A[x,y] \to B'$ det a map $g : A[x] \to B'$ and is uniquely det. by it

$\rightsquigarrow \underline{\mathrm{Der}}_A(A[x], M) := \mathrm{Hom}_{A[x]}(\Omega_{A[x]/A}, M)$.

Finally, $A \to A[x] \to B$ induces

$$\ldots \to T^0(A[x]/A, M) \to T^1(B/A[x], M) \to T^1(B/A, M) \to \cancel{T^1(A[x]/A, M)}\, 0 \to \ldots$$

<u>Known</u>: $T^0(A[x]/A, M) \simeq \mathrm{Hom}_{A[x]}(\Omega_{A[x]/A}, M)$

$T^1(B/A[x], M) \simeq \mathrm{Hom}_B(I/I^2, M)$

$\Rightarrow T^1(B/A, M) = \mathrm{coker}\left(\mathrm{Der}_A(A[x], M) \to \mathrm{Hom}_B(I/I^2, M)\right)$

$\simeq \{$ quotients $A[x,y] \to B'$ mod choise $\}$

$\simeq \{$ set of extension of B by M as A-alg $\}$

<u>Corr</u>: $B \in \underline{k\text{-alg}}$. The set of def. over $k[t]/t^2$ is in 1-1 corresp. with $T^1(B/k, B)$.