Complements and comparisons with the classical Kodaira – Spencer theory

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Goal: understand the role of the cohomology group $H^1(X, TX)$ for first order deformations.

Algebraic case

Recall: if X is a non singular affine scheme (of finite type) over k, then every deformation of X is trivial.
So we can hope to understand deformations of a non-affine scheme by using Čech cohomology over an affine ~~op~~ covering; which will give an obstruction to glueing together local trivial deformations.

Theorem. If X is a nonsingular variety (integral scheme, of finite type, separated) over k (algebraically closed). Then deformations of X over D are in bijection with $H^1(X, \mathcal{T}_X)$ where $\mathcal{T}_X = \mathcal{H}om_X(\Omega_{X/k}, \mathcal{O}_X) = \mathrm{Der}_k(\mathcal{O}_X, \mathcal{O}_X)$ is the tangent sheaf; trivial deformation corresponds to 0.

Proof. Let (U_i) open affine covering of X and X' a deformation of X over D. X is a closed subscheme of X' with the same topological space, so (U_i) induces a cover (U'_i) of X', and U'_i is a deformation of U_i over D. So it is trivial. We have isomorphisms $\varphi_i : U_i \times_k D \xrightarrow{\sim} U'_i$.
On $U_i \cap U_j$ (which is affine because X is separated) we have

$\varphi_j^{-1} \circ \varphi_i : (U_i \cap U_j) \times_k D \to (U_i \cap U_j) \times_k D$ which is an automorphism and ~~leave~~ is the identity on $U_i \cap U_j$.

dually: if $U_i \cap U_j = \mathrm{Spec}(A)$ this corresponds to

$$\theta : A \otimes D \to A \otimes D \qquad D = k[t]/t^2$$

which induces ~~an isomorphism mod t.~~ the identity mod t.

So for $x \in A \otimes D$ we can write $\theta(x) = x + t f(x)$.

Now $\theta(xy) = xy + t f(xy)$ from one side

$= \theta(x)\theta(y) = (x + t f(x))(y + t f(y))$

$= xy + t(x f(y) + y f(x))$ on the other side

so f is a derivation $A \otimes D \to A$, D-linear; this is the same as a derivation $A \to A$, k-linear.

In conclusion $\varphi_j^{-1} \circ \varphi_i$ corresponds to some $\theta_{ij} \in H^0(U_i \cap U_j, \mathcal{T}_X)$

They verify the cocycle relation

$$\theta_{ij} + \theta_{jk} + \theta_{ki} = 0$$

so they define a Čech cocycle $\theta \in H^1(X, \mathcal{T}_X)$. □

<u>example</u>: $X = \mathbb{P}^n_k$, $n \geq 1$. $H^1(X, \mathcal{T}_X) = 0$ so there are no non-trivial global deformations.

Complex analytic case

Defn a family of complex manifolds is a holomorphic map $\pi : \cancel{X} \to B$ between complex manifolds that is proper ($\forall$ compact $K \subset B$, $\pi^{-1}(K) \subset X$ is compact) and a submersion.

In this case all fibers $X_t := \pi^{-1}(t)$ are smooth compact complex manifolds of the same dimension.

Defn if Y is a compact complex manifold, a family of deformation of Y over a base B with a point $0 \in B$ is a family $\pi : X \to B$ such that $X_0 \simeq Y$.

Remark: if pt is the point, then $0 \in B$ is a map $pt \to B$ and the hypothesis is exactly that the diagram

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & X \arrow[d, "\pi"] \\ pt \arrow[r] & B \end{tikzcd}$$

is a pull-back, ie $Y = X \times_B pt$.

At any point $x \in X$ over $t \in B$, the tangent space $T_x(X_t)$ is $\mathrm{Ker}\left(d\pi(x) : T_x X \to T_t B\right)$.

We will show first that all fibres are diffeomorphic (if B is connected). So we can think of (X_t) as a family of complex manifolds which are diffeomorphic but not necessarily isomorphic; or as a family of complex structures on a fixed manifold X_0.

Then we will show how $H^1(X_0, TX_0)$ appears.

Ehresmann's theorem

Works in the $\mathcal{C}^\infty$ setting so we assume X, B are real manifolds.

Theorem. If $\pi : X \to B$ is a family of ~~manifolds~~ manifolds then locally ~~for~~ $\pi^{-1}(U) \simeq U \times X_0$ $\quad 0 \in U$

$$\begin{tikzcd} \pi^{-1}(U) \arrow[r, "\simeq"] \arrow[dr, "\pi"'] & U \times X_0 \arrow[d, "pr_U"] \\ & U \end{tikzcd}$$

The first step is to understand the ~~f~~ lifting of vector fields.
If Z is a vector field on U, a lifting is a vector field ~~on~~ $\tilde{Z}$ on $\pi^{-1}(U)$ such that $\forall x \in X$, $d\pi(x) \cdot \tilde{Z}(x) = Z(\pi(x))$.
We abbreviate this by $\pi_* \tilde{Z} = Z$
(warning: if $\tilde{Z}$ is a vector field on X there does not always exists a vector field $\pi_* \tilde{Z}$!!!).

Lemma 1. Locally around $0 \in B$, every vector field ~~can~~ Z can be lifted.

Proof:
- $0 \in B$, $f^{-1}(0)$ is compact $\subset X$.
- we can cover $f^{-1}(0)$ by a finite number of open sets V_i such that $f : V_i \to U_i \subset B$ and in some coordinates $f|V_i$ is given by a projection $(x_1, \dots, x_n) \mapsto (x_1, \dots, x_r)$ $\dim(X) = n$, $\dim(B) = r$.
- having made this choice we can easily lift a vector field on $U = \bigcap U_i$ to a vector field on some chosen $V_i \cap f^{-1}(U) := W_i$
- Now choose a partition of unity ρ_i subordinated to W_i.

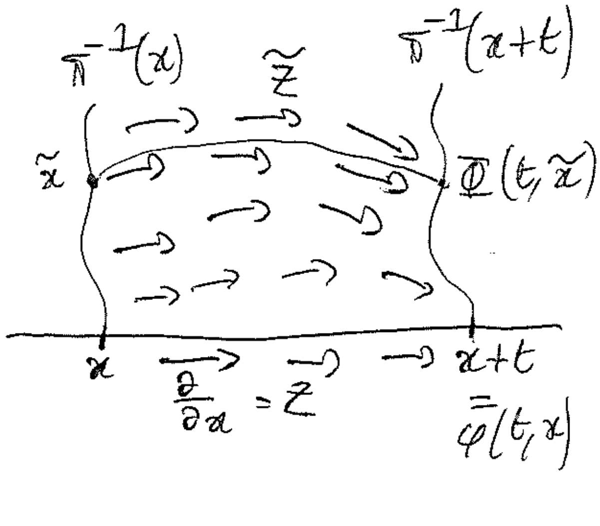
If Z is a vector field on U then we define
$\tilde{Z} = \sum \rho_i \tilde{Z}_i$ where $\tilde{Z}_i$ is a lift of Z on W_i.

Then $\pi_* \tilde{Z} = Z$. $\square$

<u>Proof of Ehresmann's theorem</u> As the result is local we can assume that we can lift vector fields on B.

- Case $\dim B = 1$: assume $B =]-1, 1[$. We write x for the coordinate on B.

On B there is a vector field $\frac{\partial}{\partial x}$ whose flow is given by $\varphi(t, x) = t + x$; ie $f(t) = t + x$ is the unique solution to the differential equation $f'(t) = \frac{\partial}{\partial x}(f(t))$ and $f(0) = x$;
starting from x the flow $\varphi(\cdot, x)$ moves x linearly at constant speed along B
Now lift $Z = \frac{\partial}{\partial x}$ to $\tilde{Z}$ and denote by $\Phi(t, \tilde{x})$ the flow of $\tilde{Z}$. Since $\tilde{Z}$ lifts Z then $\Phi(t, \tilde{x})$ "lifts" $\Phi(t, x)$ if $\tilde{x}$ lifts x: for all t, if $\pi(\tilde{x}) = x$ then $\pi(\Phi(t, \tilde{x})) = \varphi(t, x)$; and $\Phi(t, \cdot)$ induces a diffeomorphism from $\pi^{-1}(x)$ to $\pi^{-1}(t + x)$



Now if we want a map $\cancel{B \times X_0 \xrightarrow{\sim} X}$ ~~(same X~~
~~we take $x \in B$ and $y \in X_0$;~~

$$X \xrightarrow{\sim} B \times X_0 \qquad (x = \pi(\tilde{x}))\,.$$

we take $\tilde{x} \in X$, over $x \in B$. Then x is sent to 0 by the flow at time $-x$; so we send $\tilde{x}$ to $(x, \Phi(-x, \tilde{x}))$.
The map $\tilde{x} \mapsto (\pi(\tilde{x}); \Phi(-\pi(\tilde{x}), \tilde{x}))$ is $\mathcal{C}^\infty$.
In the reverse order, given $x \in B$ and $y \in X_0$, the flow at time $+x$ sends 0 to x so we send (x, y) to $\Phi(x, y)$.

- general case: assume $B =]-1, 1[^n$, write $(x_1 \dots x_n)$ the coordinates on B and $\frac{\partial}{\partial x_1} \dots \frac{\partial}{\partial x_n}$ the corresponding vector fields; $\varphi_1 \dots \varphi_n$ their flows and $\Phi_1 \dots \Phi_n$ flows of liftings.

To send $x \in B$ to 0 we first send it to $(x_1 \dots x_{n-1}, 0)$ (follow the flow of $\frac{\partial}{\partial x_n}$ for time $-x_n$); then to $(x_1 \dots x_{n-2}, 0, 0)$ (flow of $\frac{\partial}{\partial x_{n-1}}$, time $-x_{n-1}$) . . . we lift these flows:
the map $X \xrightarrow{\sim} B \times X_0$ is given by sending $\tilde{x}$ (with $\pi(\tilde{x}) = x$) to

$$\left(x; \Phi_1(-x_1, \Phi_2(-x_2, \dots \Phi_n(-x_n, \tilde{x})))\right)$$

and in reverse order $(x, y) \in B \times X_0$ is sent to

$$\Phi_1(x_1, \Phi_2(x_2, \dots \Phi_n(x_n, y)))\,.$$

Remark: if X, B are complex and π is holomorphic, it is easy to see that if we could lift holomorphic vector fields – which might not be possible because holomorphic partitions of unity doesn't exist – then $\pi^{-1}(U) \simeq U \times X_0$ would be holomorphic: 4

Assume $B = \{ z \in \mathbb{C} / \ |z| < 1 \}$, on B we have the holomorphic vector field $Z = \frac{\partial}{\partial z}$ whose flow at time $t \in \mathbb{C}$ is given by $\varphi(t, z) = t + z$. If we lift Z to $\tilde{Z}$ then $\tilde{Z}$ has a flow Φ which is holomorphic; and

$\tilde{z} \mapsto (\pi(\tilde{z}); \Phi(-\pi(\tilde{z}), \tilde{z}))$ is a biholomorphism

$$X \xrightarrow{\sim} B \times X_0 .$$

Kodaira – Spencer class and map.

Suppose $\pi: X \to B$ is a family of compact complex manifolds with $0 \in B$.

We have an exact sequence of vector bundles over X_0

$$0 \to TX_0 \to TX|_{X_0} \xrightarrow{d\pi} \pi^* TB \to 0$$

and $\pi^* TB \simeq T_0 B$ (as a trivial vector bundle)

because $\forall x \in X_0$, $\pi(x) = 0$.

This sequence induces in cohomology a map. ← holomorphic tangent space

$$H^0(X_0, \pi^* TB) \simeq T_0 B \to H^1(X_0, TX_0)$$

called the Kodaira – Spencer map. We can think of it as the derivative of the family at 0 in a direction given in $T_0 B$.

One possible interpretation of $H^1(X_0, TX_0)$: in the family $\pi: X \to B$, it is easy to find a finite number of open sets U_i of B $(0 \in U_i)$, V_i of X which cover X_0 such that we can lift holomorphic vector fields on U_i to V_i (same argument as the real case — we do not use partition of unity).

Take $U = \bigcap U_i$, $W_i = \pi^{-1}(U) \cap V_i$, take Z a holomorphic vector field on U, Z_i a lift to W_i, holomorphic.

On $W_i \cap W_j$ we have $\pi_* Z_i = \pi_* Z_j = Z$ so $Z_i - Z_j$ is in $\mathrm{Ker}(d\pi)$, that is $TX_0|_{W_i \cap W_j}$!!

So the $Z_i - Z_j$ define a Čech cycle in $H^1(X_0, TX_0)$.

If we change the holomorphic liftings Z_i to some $Z_i + A_i$ then on X_0: $\pi_*(Z_i + A_i) = \pi_*(Z_i) = Z$ so $A_i \in \mathrm{Ker}\, d\pi$, A_i is a holomorphic vector field on W_i in TX_0.

and so $(Z_i + A_i) - (Z_j + A_j) = (Z_i - Z_j) + (A_i - A_j)$ with $A_i - A_j$ a 1-Čech-coboundary; so the class of the $Z_i - Z_j$ in $H^1(X_0, TX_0)$ does not depend on the choice of Z_i.

We conclude that if $H^1(X_0, TX_0) = 0$ then every holomorphic vector field Z on B can be (locally) lifted to X_0 holomorphically, and so we can prove that every deformation of X_0 is trivial.

In fact what we have defined depends only on $Z(0 \in B)$ so it is a map $T_0 B \to H^1(X_0, TX_0)$, the Kodaira–Spencer map, and a map $H^0(U, TB) \to H^1(\pi^{-1}(U), T_{X/B})$ where $T_{X/B}$ is the relative tangent bundle $= \mathrm{Ker}(d\pi: TX \to \pi^* TB)$ $=$ holomorphic vector fields Y such that $\pi_* Y = 0$.