Higher order deformations

Louis-Clément Lefèvre

Recall on Artin rings

We fix k: field, characteristic 0, algebraically closed.
and C, C' : Artin local k-algebras.

<u>Def:</u> An <u>elementary extension</u> is an exact sequence
$$0 \to J \to C' \xrightarrow{\varphi} C \to 0$$
where J is an ideal of C' such that, if $m_{C'}$ is the maximal ideal of C', $J \cdot m_{C'} = 0$.

In particular $J^2 = 0$ $(J \subset m_{C'})$.
This implies that J is a C-module, take
$x \in J$, $a \in C$. Lift a to $b \in C'$ $(\varphi(b) = a)$
and define $a \cdot x := b \cdot x$.
if we choose another lifting $c \in C'$ then
$b - c \in J$ so $(b-c) \cdot x = 0$ so $b \cdot x = c \cdot x$.

The extension is <u>split</u> if it is isomorphic to $J \oplus C$ as extension, ie there is a commutative diagram

$$\begin{tikzcd} & & C' \arrow[d, "\wr"] \arrow[dr] & & \\ 0 \arrow[r] & J \arrow[ur] \arrow[dr] & {} & C \arrow[r] & 0 \\ & & J \oplus C \arrow[ur] & & \end{tikzcd}$$

In a split extension there is a canonical way to lift an element of C to C'.

Basic example of elementary extension:

$$0 \to (t^{n+1}) \to k[t]/(t^{n+2}) \to k[t]/(t^{n+1}) \to 0$$

$$0 \to (t) \to k[t]/(t^2) \to k \to 0$$

the last one splits, ~~te~~

Theorem: every Artin local k-algebra can be obtained as successive elementary extensions; every surjection $C' \to C$ is a composition of elementary extensions.

Proof. take C, m_C m_C is nilpotent, $m_C^n = 0$.

Then $0 \to m_C^{n-1} \to \underset{\overset{\shortparallel}{C}}{C/m_C^n} \to C/m_C^{n-1} \to \cdots \to C/m_C^2 \to \underset{\overset{\shortparallel}{k}}{C/m_C} \to 0$

- if $C' \to C$ is surjective, take J the kernel and $n = \max\{n \mid m_{C'}^n J = 0\}$, then

$$0 \to m_{C'}^{n-1} \cdot J \to \underset{\overset{\shortparallel}{C'}}{C'/m_{C'}^n \cdot J} \to C'/m_{C'}^{n-1}J \to \cdots \to C'/m_{C'} \cdot J \to \underset{\overset{\shortparallel}{C}}{C'/J} \to 0$$

So we have to understand deformations over an Artin ring by deforming successively over elementary extensions; We fix the notation $0 \to J \to C' \to C \to 0$.

Recall on deformations

Algebraic case: we fix X_0: scheme $/k$

A deformation of X_0 over C is a scheme X, flat over C, with a closed immersion $i: X_0 \to X$

such that

$$\begin{tikzcd} X_0 \arrow[r, "i"] \arrow[d] & X \arrow[d] \\ \mathrm{Spec}(k) \arrow[r] & \mathrm{Spec}(\mathbb{C}) \end{tikzcd}$$

is a pullback diagram ($X_0 \simeq X \times_{\mathbb{C}} k$).

Remark: given a map $i : X_0 \to X$, this implies that i is a closed immersion.

<u>Differential case</u>: X_0 is a manifold, pt is the point (as 0-dimensional manifold). There is a unique map $X_0 \to pt$.

Let S be a manifold with a base-point s, seen as a map $s : pt \to S$.

A deformation of X_0 over S is a family $\pi : X \to S$ (X is a manifold, π is proper and ~~surj~~ a submersion) with a given embedding $f : X_0 \to X$ such that $f(X_0) \simeq \pi^{-1}(s)$. With diagram this means exactly

$$\begin{tikzcd} X_0 \arrow[r, "f"] \arrow[d] & X \arrow[d, "\pi"] \\ pt \arrow[r, "s"] & S \end{tikzcd}$$

is a pullback.

So we should work and think with diagrams; This gives automatically definitions for isomorphisms of deformations —

In both cases we have the <u>trivial</u> deformation:

algebraic case: $X_0 \hookrightarrow X_0 \times_k C$

~~analytic~~ differential case: $X_0 \overset{\text{over } \Delta}{\hookrightarrow} X_0 \times S$

and a deformation is called trivial if it is isomorphic to the trivial deformation.

<u>Definition</u>: given $0 \to J \to C' \to C \to 0$ and a deformation X of X_0 ~~over~~ over C, an <u>extension</u> of the deformation is a scheme X' over C', with a closed immersion $j: X \to X'$ such that

$$\begin{array}{ccccc} X_0 & \xrightarrow{i} & X & \xrightarrow{j} & X' \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Spec}(k) & \to & \mathrm{Spec}(C) & \to & \mathrm{Spec}(C') \end{array}$$

is composed of 3 pullback diagrams.

Basic example: ~~take~~ fix X_0, take $X = X_0 \times_k C$ and $X' = X_0 \times_k C'$ (trivial).

<u>Problem</u>: - given X, an extension X' might not exist, even trivially !!!

- there is no $X \times_C C'$ because C' is not a C algebra.
- to construct X' locally ($X = \mathrm{Spec}(A)$, $X' = \mathrm{Spec}(A')$) we should lift a C-algebra A to a C'-algebra A'. This might be not possible, and if it is, not canonical (unless the sequence $J \to C' \to C$ splits).
- if it ~~this~~ is possible to construct X' locally, then globally we should glue extensions;

but this might not be possible ~~to glue~~ ~~globally~~ and then should be an obstruction, measured by a Čech cocycle and related to ~~the~~ the different local liftings possible.

Two express the "non-canonical choice" we need this notion.

Definition If a group G acts on a set S, we say S is a principal homogeneous space, or torsor, $\Leftrightarrow$ G acts freely and transitively $\Leftrightarrow$ for all $s_0 \in S$ $\begin{array}{l} G \to S \\ g \mapsto g.s \end{array}$ is a bijection.

So we can think of S as something non-canonically in bijection with G.

We say pseudotorsor if S can be empty.

Think of an affine space $\mathcal{E}$ with an action of a vector space $(E,+)$: $\mathcal{E}$ is like E without a particular choice of an origin.

Think of the

Theorem: if $f: E \to F$ is a linear map between vector spaces and if $y \in F$ then

- either $f^{-1}(y) = \emptyset$
- or there exists $x \in f^{-1}(y)$ and

$$f^{-1}(y) = \{ x + u \mid u \in \operatorname{Ker} f \}$$

So $f^{-1}(y)$ is a pseudotorsor for $\operatorname{Ker} f$; there is no canonical x.

(we use this theorem to solve linear differential equation and write the homogeneous equation associated |

General picture

Remark also that in $J \to C' \to C$, if $x \in C$, the set of liftings of x to C' is a torsor for J.

Suppose we fix $0 \to J \to C' \to C \to 0$ and we fix X_0, X, X' (we can take the trivial $X = X_0 \times_k C$ and $X' = X_0 \times_k C'$).

Suppose we have some structure on X (line bundle, coherent sheaf, closed subscheme) and a deformation on X. We want to study how it extends to X'. It needs to lift ~~C-algebras (or modules)~~ ~~to~~ elements of C-algebras (or modules) to elements of C'-algebras (or modules)

- Locally, it is not always possible
- Locally, when it is possible to lift, it is not canonical. The set of liftings is a torsor under some Čech cohomology group that depends on X and J.
- Globally, there is an obstruction to glue the local extensions. The liftings must coincide, and so the global obstruction is the next Čech cohomology group

Line bundles

Suppose $\mathcal{L}_0$ is a line bundle on X_0, $\mathcal{L}$ is a deformation on X and we look for extensions $\mathcal{L}'$ on X'.

~~(so there are fixed maps $i: \mathcal{L}_0 \to \mathcal{L}$, $j: \mathcal{L} \to \mathcal{L}'$;~~
~~recall that X_0, X, X' have the same topological space;~~

Recall that X_0, X, X' have the same underlying topological spaces; this means that there are map $\mathcal{L} \to \mathcal{L}_0$ and $\mathcal{L}' \to \mathcal{L}$ that induces isomorphisms

$$\mathcal{L} \underset{\mathcal{O}_X}{\otimes} \mathcal{O}_{X_0} \xrightarrow{\sim} \mathcal{L}_0 \quad \text{and} \quad \mathcal{L}' \underset{\mathcal{O}_{X'}}{\otimes} \mathcal{O}_X \xrightarrow{\sim} \mathcal{L}.$$

Note that if $X = X_0 \times_k C$ then $\mathcal{O}_X = \mathcal{O}_{X_0} \otimes_k C$

and if $X' = X_0 \times_k C'$ then $\mathcal{O}_{X'} = \mathcal{O}_{X_0} \otimes_k C'$.

Theorem. i. There exists a map $\delta : \{\text{lines bundles on } X\} \to H^2(X, J \otimes \mathcal{O}_X)$ such that $\delta(\mathcal{L}) = 0 \iff$ an extension $\mathcal{L}'$ exists over X'.

ii. If such an $\mathcal{L}'$ exists, $H^1(X, J \underset{C}{\otimes} \mathcal{O}_X)$ acts transitively on isomorphism classes of such $\mathcal{L}'$.

iii. This set is a torsor $\iff$ the natural map

$$H^0(X, \mathcal{O}_{X'}^*) \to H^0(X, \mathcal{O}_X^*) \quad \text{is surjective}$$

(note: this can be seen as $\mathrm{Aut}(\mathcal{L}') \to \mathrm{Aut}(\mathcal{L})$)

iv. This holds if $H^0(X, \mathcal{O}_{X_0}) = k$

Proof i. On X there is an exact sequence of $\mathcal{O}_X$-modules

$$0 \to J \otimes \mathcal{O}_X \to \mathcal{O}_{X'} \to \mathcal{O}_X \to 0$$

We replace it by

$$0 \to J \otimes \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_{X'}^* \to \mathcal{O}_X^* \to 0$$

where exp is a truncated exponential map $x \in J \mapsto (1+x)$

Then take the long exact sequence

$$0 \to H^0(X, J\otimes O_X) \to H^0(X', O_{X'}^*) \xrightarrow{a} H^0(X, O_X^*)$$
$$\xrightarrow{b} H^1(X, J\otimes O_X) \xrightarrow{c} H^1(X, O_{X'}^*) \xrightarrow{d} H^1(X, O_X^*)$$
$$\xrightarrow{\delta} H^2(X, J\otimes O_X) \to \cdots$$

It is then quite clear: $\mathcal{L}$ lives in $H^1(X, O_X^*)$ and we want to lift it by d. It is possible $\Leftrightarrow$ $\delta(\mathcal{L}) = 0$.
If $\mathcal{L}'$ exists, ~~the sets~~ it is defined modulo $\mathrm{Im}(c)$.
So it is a torsor under $H^1(X, J\otimes O_X)$ $\Leftrightarrow$ c is injective
$\Leftrightarrow$ $b = 0$ $\Leftrightarrow$ a is surjective.
We admit point iv.

<u>Exercise:</u> Understand the proof in case $X = X_0 \times C$, $X' = X_0 \times C'$ by writing explicit cocycles, and convince yourself how it fits in the general picture.
To do: clarify the functoriality, write i^*, i^{-1} where $i: X_0 \hookrightarrow X$.

<u>Closed subschemes.</u>

We fix X_0, X, X', suppose Y_0 is a closed subscheme of X_0 and Y a deformation on X, ie a closed subscheme $Y \subset X$ such that $Y \times_C k = Y_0$. We look for extensions Y' to X', ie closed subschemes with $Y' \times_{C'} C = Y$; we have diagrams

$$\begin{tikzcd} Y_0 \arrow[r, hook] \arrow[d] & Y \arrow[r, hook] \arrow[d] & Y' \arrow[d] \\ X_0 \arrow[r, hook] \arrow[d] & X \arrow[r, hook] \arrow[d] & X' \arrow[d] \\ k \arrow[r] & C \arrow[r] & C' \end{tikzcd}$$

where $\hookrightarrow$ denotes closed subschemes and we see pullbacks over $k \to C$ and $C \to C'$.

Theorem. • The set of extensions of Y over C' is a pseudotorsor under the action of $H^0(Y_0, N_0 \otimes_k J)$ where N_0 is the normal sheaf N_{Y_0/X_0}.

• If extensions exists locally on X (technical hypothesis; we admit it is true if $Y_0 \subset X_0$ is a local complete intersection; or if Y_0, X_0 are smooth). Then there is an obstruction $\alpha \in H^1(Y_0, N_0 \otimes_k J)$ such that $\alpha = 0$ $\Leftrightarrow$ Y' exists globally. Then the set of all extensions is a torsor under $H^0(Y_0, N_0 \otimes_k J)$.

The proof is clear in the book. It begins by the first part, locally then globally; the pseudotorsor corresponds to different possible liftings. Then we use it to prove the second part, where we must make compatible local liftings: the obstruction is measured by a Čech cohomology class of degree 1.

Corollary. If $X_0 = \mathbb{P}^n_k$ and Y_0 is a closed subscheme, assume there are no local obstructions to deformation of Y_0. Let y be the point of the Hilbert scheme H corresponding to Y. Then H is nonsingular at y.

Proof. We use the infinitesimal ~~property~~ lifting property. We must show that for all C and $f: \mathrm{Spec}(C) \to H$ that sends the unique closed point to y, and for any surjection $C' \to C$, f lifts to $g: \mathrm{Spec}(C') \to H$. We can suppose $C' \to C$ is an elementary extension.

By the functorial property of H, f corresponds exactly to a deformation of Y_0 over C and g to an extension of the deformation ~~to~~ over C'.

(If T is a scheme: $\mathrm{Hom}(T, H) \Leftrightarrow$ closed subscheme $\subset X \times T$, flat over T. ~)

If extensions of deformations exist, then f lifts to g.
So y is nonsingular.