Obstructions to Deformations of Schemes

Recall: X_0 scheme$/k$, C Artin local ring$/k$

A deformation of X_0 over C is a scheme $\mathfrak{X}$ flat $/C$ with $X_0 \hookrightarrow \mathfrak{X}$ (closed immersion) st $X_0 \simeq \mathfrak{X} \times_C k$.

Def: X_0 scheme$/k$, $C' \twoheadrightarrow C$ map between Artin loc. rings$/k$

Then,
$$\begin{tikzcd} X_0 \arrow[r, hook] \arrow[d] & \mathfrak{X}' \arrow[d] \\ \operatorname{Spec} k \arrow[r, hook] & \operatorname{Spec} C' \end{tikzcd}$$
extends
$$\begin{tikzcd} X_0 \arrow[r, hook] \arrow[d] & \mathfrak{X} \arrow[d] \\ \operatorname{Spec} k \arrow[r, hook] & \operatorname{Spec} C \end{tikzcd}$$
if there exists

$$\begin{tikzcd} X_0 \arrow[rr, hook] \arrow[dr, hook] \arrow[dd] & & \mathfrak{X}' \arrow[dd] \\ & \mathfrak{X} \arrow[ur, hook] \arrow[dd] & \\ \operatorname{Spec} k \arrow[rr, hook] \arrow[dr, hook] & & \operatorname{Spec} C' \\ & \operatorname{Spec} C \arrow[ur, hook] & \end{tikzcd}$$

commutative st

$$\mathfrak{X} \xrightarrow{\sim} \mathfrak{X}' \times_{C'} C$$

Two extensions are equivalent if $\exists$ isom$|_C$ compatible with the immersions of X_0.

Question: Given
$$\begin{tikzcd} X_0 \arrow[r, hook] \arrow[d] & \mathfrak{X} \arrow[d] \\ \operatorname{Spec} k \arrow[r] & \operatorname{Spec} C \end{tikzcd}$$
and $C' \twoheadrightarrow C \to 0$

(1) $\exists$ an extension?

(2) How many (mod. equivalence)?

Affine case (local): $X_0 = \operatorname{Spec} B_0$, $B_0 = k[x_1, \dots, x_n]/I_0$

$\mathfrak{X} = \operatorname{Spec} B$, $B = C[x_1, \dots, x_n]/I$

Potential deformations of $C' \twoheadrightarrow C$ (fixed) are $\mathfrak{X}' = \operatorname{Spec} B'$

$B' = C'[x_1, \dots, x_n]/I'$

Remark: $r = m$ Look [Vistoli] page 5

(eg, $I = (f_1, \dots, f_r)$ and $I' = (f_1', \dots, f_m')$)

~~[illegible]~~ Whether can we make B' flat over C'?

Let's consider $0 \to J \to C' \to C \to 0$, $J^2 = 0$. $(*)$

[Prop 2.2 pag 10 Hartshorne]:

$M' \in C'\text{-mod}$ is flat$/C' \iff M = M' \otimes_{C'} C$ flat $/C$

and $M \otimes_C J \hookrightarrow M'$.

Let: $R = C[x_1, \dots, x_n]$, $R' = C'[x_1, \dots, x_n]$, $R_0 = k[x_1, \dots, x_n]$

Relations: $0 \to Q \to R^r \xrightarrow{f} I \to 0$,
(Q & Q') $0 \to Q' \to R'^r \xrightarrow{f'} I' \to 0$,

Tensoring $(*) \otimes_{C'}$ with R'^r, R', and B':

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & Q' \arrow[r, "\phi"] \arrow[d] & Q \arrow[d] \arrow[ddddlll, dashed, bend left, "\delta_0"] & \\
0 \arrow[r] & R_0^r \otimes_k J \arrow[r] \arrow[d] & R'^r \arrow[r] \arrow[d, "f'"] & R^r \arrow[r] \arrow[d, "f"] & 0 & (*) \otimes_{C'} R'^r \\
0 \arrow[r] & R_0 \otimes_k J \arrow[r] \arrow[d] & R' \arrow[r] \arrow[d] & R \arrow[r] \arrow[d] & 0 & (*) \otimes_{C'} R' \\
 & B_0 \otimes_k J \arrow[r] \arrow[d] & B' \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & 0 & (*) \otimes_{C'} B' \\
 & 0 & 0 & 0 & &
\end{tikzcd}$$

Relations

Lifting of relations

B' is flat C'-mod $\iff B_0 \otimes_k J \hookrightarrow B'$ [Prop 2.2]
(ie, a deformation)

Snake lemma: $\exists\, \delta_0 : Q \to B_0 \otimes_k J$ st

$B_0 \otimes_k J \hookrightarrow B' \iff \delta_0 = 0$.

Remark: Any element in $F_0 \subseteq Q$ goes to zero by δ_0;
where $F_0 = \langle f(a)b - f(b)a \rangle_{a,b \in R^r}$ corresponds to the
Koszul relations $(0, \dots, 0, -f_j, \dots, f_i, 0, \dots, 0) \xmapsto{f} -f_j \cdot f_i + f_i \cdot f_j = 0$
that we can always lift, to $-f_j' f_i' + f_i' f_j' = 0$.
$(\operatorname{Im} \phi = \ker \delta_0)$

So we have $\delta_1 \in \mathrm{Hom}_{C'}(Q/F_0, B_0 \otimes_k J)$

$Q/F_0 \to R^r/IR^r$ allows us to define

$\delta \in \mathrm{coker}\,(\mathrm{Hom}(R^r/IR^r, B_0 \otimes J) \to \mathrm{Hom}(Q/F_0, B_0 \otimes J))$

$$\overset{\text{def}}{=} T^2(B/C; B_0 \otimes J) \underset{\text{base change}}{\cong} T^2(B_0/k; B_0 \otimes J)$$

Rem: δ indep. of choises (depends only on $\begin{matrix} & B \\ & \downarrow \\ C' \twoheadrightarrow & C \end{matrix}$)

Now: If B'/C' exists $\underset{\text{flatness}}{\Longrightarrow} \delta_0 = 0 \Rightarrow \delta = 0$ ✓

If $\delta = 0$: By definition, $\delta_1 \in \mathrm{Hom}(Q/F_0, B_0 \otimes J)$ lifts to $\gamma: R^r/IR^r \to B_0 \otimes_k J$.

$$\Rightarrow R^r \nearrow R^r/IR^r \searrow B_0 \otimes_k J, \quad R^r \longrightarrow B_0 \otimes_k J$$

R^r free ($\Rightarrow$ projective):

$$\begin{tikzcd} R_0 \otimes_k J \arrow[r, two heads] & B_0 \otimes_k J \arrow[r] & 0 \\ & R^r \arrow[ul, dashed, "\exists \tilde{\gamma}"] \arrow[u] & \end{tikzcd}$$

$\tilde{\gamma}$ defined by $g_1, \dots, g_r \in R_0 \otimes_k J$

Take $\tilde{I} = (f_i' - g_i') \rightsquigarrow \tilde{B} = R'/\tilde{I} \rightsquigarrow \tilde{\delta}_0$ as before

We compute $\tilde{\delta}_0 = 0 \Rightarrow \tilde{B}$ flat over C'. ✓

We proved: $\exists\, \delta \in T^2(B_0/k; B_0 \otimes J)$ (obstruction) st $\delta = 0 \iff \exists\, B'$ extension of B.

Previous seminar: Deformations are a torsor under $\mathrm{Hom}(I/I^2, B \otimes J)$

Seen: [Lemma 4.5] Different choises are parametrized by $\mathrm{Hom}(\Omega_{R/C}, B \otimes J)$

So, if $\exists$ extensions, the set of equiv. classes is a torsor under

$\mathrm{coker}(\mathrm{Hom}(\Omega_{R/C}, B \otimes J) \to \mathrm{Hom}(I/I^2, B \otimes J)) := T^1(B/C; B \otimes J) \underset{\text{base change}}{\cong} T^1(B_0/k; B_0 \otimes$

Seen [Lemma 4.5]: Group of automorphisms of B' lifting id_B is isomorphic to $T^0(B_0/k; B_0 \otimes J)$.

Rmk: 1) If X_0 l.c.i. $\Leftrightarrow Q = F_0$
$\Rightarrow \mathrm{Hom}(Q/F_0, B_0) = 0 \Rightarrow T^2 = 0$: no obstruction!

2) X_0 scheme/k. We can consider $T^i_{X_0}$ as sheaves defined by affine charts. By [Schlessinger, "On rigid singularities" page 150]: If X_0 is normal, reduced of positive dimension:

$$\mathrm{Ext}^1_{\mathcal{O}_{X_0}}(\Omega_{X_0}, \mathcal{O}_{X_0}) \xrightarrow{\sim} T^1_{X_0} \quad (:= T^1_{X_0}(X_0/\mathrm{Spec}\, k; \mathcal{O}_{X_0}))$$

$$\mathrm{Ext}^2_{\mathcal{O}_{X_0}}(\Omega_{X_0}, \mathcal{O}_{X_0}) \xrightarrow{\sim} T^2_{X_0}$$

Now, consider the global situation: X_0 scheme / k and ask the same questions for $0 \to J \to C' \to C \to 0$, $J^2 = 0$.

Thm:

(Obs) (a) $\exists$ 3 successive obstructions ~~lying~~ for the existence of $\mathcal{X}'$ extension of $\mathcal{X}$ over C', lying on $H^0(X_0, T^2_{X_0} \otimes J)$, $H^1(X_0, T^1_{X_0} \otimes J)$ and $H^2(X_0, T^0_{X_0} \otimes J)$.

(Def) (b) Let $\mathrm{Def}(\mathcal{X}/C; C') = \dfrac{\text{extensions } \mathcal{X}'/C'}{\text{mod. equivalence}}$. Then,

$$0 \to H^1(X_0, T^0_{X_0} \otimes J) \to \mathrm{Def}(\mathcal{X}/C; C') \to H^0(X_0, T^1_{X_0} \otimes J)$$
$$\to H^2(X_0, T^2_{X_0} \otimes J) \quad \text{is exact.}$$

(Aut) (c) Given $\mathcal{X}'/C'$ extension of $\mathcal{X}/C$, $H^0(X_0, T^0_{X_0} \otimes J)$ is the group of automorphisms of $\mathcal{X}'/C'$ lifting the identity of $\mathcal{X}/C$.

Proof: (a) • Write $\mathcal{X} = \bigcup \mathcal{U}_i$, $\mathcal{U}_i$ affine
$\rightsquigarrow$ Obs in $H^0(\mathcal{U}_i, T^2_{\mathcal{U}_i} \otimes J)$ for $\exists\, \mathcal{U}_i'$ extension
$\rightsquigarrow \delta_1 \in H^0(X_0, T^2_{X_0} \otimes J)$ global obstruction.

• If $\delta_1 = 0$, $\exists\, \mathcal{U}_i'$ extension. For $\mathcal{U}_{ij} = \mathcal{U}_i \cap \mathcal{U}_j$
$\rightsquigarrow$ 2 extensions: $\mathcal{U}_i'|_{\mathcal{U}_{ij}}$ and $\mathcal{U}_j'|_{\mathcal{U}_{ij}} \leftrightarrow$ 2 elements in $H^0(\mathcal{U}_{ij}, T^1_{\mathcal{U}_{ij}} \otimes J)$

Their difference δ_{ij} belongs to $H^0(\mathcal{U}_{ij}, T^1_{\mathcal{U}_{ij}} \otimes J)$

$\mathcal{U}_{ij} \rightsquigarrow \delta_{ij}$, $\mathcal{U}_{ik} \rightsquigarrow \delta_{ik}$, $\mathcal{U}_{jk} \rightsquigarrow \delta_{jk}$

By ~~definition~~ definition of δ_{ij}: $\delta_{ij} - \delta_{ik} + \delta_{jk} = 0$

$\to \delta_2 \in H^1(X_0, T^1_{X_0} \otimes J)$ obstruction.

- If $\delta_2 = 0$: The extensions $\mathcal{U}_i'$ are equiv. on $\mathcal{U}_{ij}$
  $\to \varphi_{ij}: \mathcal{U}_i'|_{\mathcal{U}_{ij}} \xrightarrow{\sim} \mathcal{U}_j'|_{\mathcal{U}_{ij}}$ isom. ; induces an automorphism of $\mathcal{U}_i'|_{\mathcal{U}_{ijk}} \longleftrightarrow H^0(\mathcal{U}_{ijk}, T^0_{\mathcal{U}_{ijk}} \otimes J)$
  They agree in 4-intersections $\rightsquigarrow \delta_3 \in H^2(X_0, T^0_{X_0} \otimes J)$.

(b) Given $\mathcal{X}'/C'$ and $\mathcal{X} = \bigcup \mathcal{U}_i$, $\mathcal{U}_i$ affine
$\to$ element in $H^0(\mathcal{U}_i, T^1_{\mathcal{U}_i} \otimes J) \to$ element in $H^0(X_0, T^1_{X_0} \otimes J)$

But given an element in $H^0(X_0, T^1_{X_0} \otimes J)$
$\to$ give extensions $\mathcal{U}_i'$ for each $\mathcal{U}_i$ and the obstruction to glue together along $\mathcal{U}_{ij}$ lies in an element in $H^2(X_0, T^0_{X_0} \otimes J)$ (that must vanish) $\to$ Im = ker

$$\Rightarrow \mathrm{Def}(\mathcal{X}'/C'; C) \to H^0(X_0, T^1_{X_0} \otimes J) \to H^2(X_0, T^0_{X_0} \otimes J)$$

- If $\mathcal{X}_1', \mathcal{X}_2'$ give the same element in $H^0(X_0, T^1_{X_0} \otimes J)$
  $\Rightarrow \forall \mathcal{U}_i$, $\varphi_i: \mathcal{X}_1'|_{\mathcal{U}_i} \xrightarrow{\sim} \mathcal{X}_2'|_{\mathcal{U}_i} \to \psi_{ij} = \varphi_j^{-1} \circ \varphi_i$ aut. of $\mathcal{X}_1'|_{\mathcal{U}_{ij}} \to$ element in $H^0(\mathcal{U}_{ij}, T^0_{\mathcal{U}_{ij}} \otimes J)$
  They agree in 3-intersections: element in $H^1(X_0, T^0_{X_0} \otimes J)$
  If this element is zero in $H^1(X_0, T^0_{X_0} \otimes J)$ the automorp. coincide in the $\mathcal{U}_{ij} \Longrightarrow \mathcal{X}_1', \mathcal{X}_2'$ globally isomorphic ∎

<u>Corollary</u>: If X_0 is not singular, then
a) $\exists$ only one obstruction in $H^2(X_0, T_{X_0} \otimes J)$ for the existence of $\mathcal{X}'/C'$.
b) If the extensions exist, their equiv. classes form a torsor under $H^1(X_0, T_{X_0} \otimes J)$.

<u>Proof</u>: X_0 non singular $\Rightarrow T^1_{X_0} = T^2_{X_0} = 0$ and
$T^0_{X_0} = T_{X_0}$ (tangent sheaf) ∎