Plane curve singularities

"Local moduli": Let's focus on a single plane singularity in order to try to describe all possible local deformations of the singularity.

Goal: Find $\begin{array}{c} \mathcal{X} \\ \downarrow \\ \operatorname{Spec} R \end{array}$ deformation

being

-) "complete" (versal): any other local def. can be obtained (up to isom.) by base extension from this one.
-) "minimal" (miniversal): it is the smallest possible
-) "universal": any other family is obtained by unique base ~~change~~ extension from this one.

Exemple: Consider $(xy=0) \subseteq \mathbb{A}^2 = \operatorname{Spec} k[x,y]$

and $\mathcal{X}_t = (xy-t=0) \subseteq \mathbb{A}^3 = \operatorname{Spec} k[x,y,t]$

$$\begin{array}{c} \downarrow \\ T = \operatorname{Spec} k[t] \end{array}$$

$t \neq 0 \Rightarrow \mathcal{X}_t$ hyperbola.

$t = 0 \Rightarrow \mathcal{X}_0$ node.

Let $\mathcal{X}'/S$ be any deformation of the node st $S = \operatorname{Spec} \hat{R}$, $\hat{R}$: complete local ring

For simplicity: We will assume

- $\hat{R} = k[[s]]$
- $\mathcal{X}'$ defined by one equation $g(x,y,s) = 0$, $g \in k[[s]][x,y]$ with $g(x,y,0) = xy$.

Aim: there exist a morphism $S \to T$ (ie, $k[t] \xrightarrow{\varphi} k[[s]]$ given by a power series $\varphi(t) = T(s)$; $T(0) = 0$) st $\mathcal{X}' \simeq \mathcal{X} \times_T S$

To have $\mathcal{X}' \simeq \mathcal{X} \times_T S$ it will sufficient to find $X(x,y,s)$, $Y(x,y,s)$, $U(x,y,s) \in k[[s]][x,y]$

st
- $X(x,y,0) = x$
- $Y(x,y,0) = y$
- $U(x,y,s)$ unit st $U(x,y,0) = 1$

and $U(XY - T) = g(x,y,s)$ $(*)$

Construction:

Datum: $T(0) = 0$, $X(0) = x$, $Y(0) = y$, $U(0) = 1$,

$$g = xy + \sum_{i \geq 1} g_i s^i, \quad g_i \in k[x,y].$$

Let: $T = \sum_{i \geq 1} a_i s^i$

$$X = x + \sum_{i \geq 1} b_i s^i, \quad Y = y + \sum_{i \geq 1} c_i s^i$$

$$U = 1 + \sum_{i \geq 1} u_i s^i \quad ; \; a_i \in k, \; b_i, c_i, u_i \in k[x,y].$$

Substituting and looking at the deg 1 part of $(*)$ (coeff of s) we obtain:

$$U(XY - T) = g(x, y, s)$$

$$(1 + u_1 s + \dots)\big((x + b_1 s + \dots)(y + c_1 s + \dots) - a_1 s - \dots\big) = xy + g_1 s + \dots$$

$$xy + \underline{b_1 y s} + \underline{x c_1 s} - \underline{a_1 s} + \underline{u_1 xys} + \dots = xy + \underline{g_1 s} + \dots$$

$$\Rightarrow \boxed{y b_1 + x c_1 - a_1 + xy u_1 = g_1} \quad (*)_1$$

given

$\Rightarrow$ We can solve $(*)_1$ and a_1 is <u>uniquely determined</u>!

(but b_1, c_1, u_1 are <u>not</u> unique).

<u>By induction</u>: suppose a_i, b_i, c_i, u_i have been chosen for $i < n$ (ie, $(*)_i$ solved for $i < n$)

For s^n:

$$h(x,y) + x c_n + y b_n - a_n + xy u_n = g_n \quad (*)_n$$

all crossing products ~~involving~~ with $i < n$ (determined)

$\Rightarrow$ We can solve $(*)_n$

We find T, X, Y, U <u>but</u> they belong to $k[x,y][[s]]$ (bigger than $k[[s]][x,y]$!)

What do we have?

Recall: Let X be a noetherian scheme and $Y \subseteq X$ closed subscheme. The <u>formal completion</u> of X along Y, denoted $(\hat{X}, \mathcal{O}_{\hat{X}})$ is the following ringed space:

- Topological space: Y
- Sheaf of rings: $\mathcal{O}_{\hat{X}} = \varprojlim \mathcal{O}_X / \mathcal{I}_Y^n$

where $\mathcal{I}_Y$ is the sheaf of ideals defining Y.

<u>Here</u>: The <u>formal completions</u> of $\mathcal{X}'$ and $\mathcal{X} \times_T S$ along the closed fiber at $s = 0$ are isomorphic.

<u>So</u>, the versality property is true only in this formal sense.

<u>Remarks</u>:

1) Everything works if we consider power series in x and y instead of polynomial, we obtain an analogous versal deformation property.
2) The linear coeff. a_1 is uniquely determined, so $S \to T$ induces a <u>unique</u> map on Zariski tangent spaces $\Rightarrow T$ is small as possible, i.e., it is a miniversal deformation space for $(xy = 0)$
3) a_i, $i > 1$ are not unique: Let $u \in k[[s]]^*$ with constant term 1.

$u^{-1}((xu)y - su) = xy - s$

$\rightarrow$ we get an isomorphism of $\mathcal{X}_t$ by taking

$U = u^{-1}$, $X = xu$, $Y = y$, $T = su$

Set $u = 1 - s$, $u^{-1} = 1 + s + s^2 + \cdots$

$\Rightarrow T = s - s^2 \Rightarrow a_2 = -1$

while for the trivial isom, $a_2 = 0$.

So, $S \to T$ is <u>not</u> unique. (the def. is <u>not</u> universal)

4) We could take $u_n = 0$ $\forall n \geq 1$ in the above proof.
We include U because it becomes necessary in the general case when $f(x,y) = 0$ is not homogeneous.

Assume $f(x,y) = 0$ isolated sing. at the origin.

$\Rightarrow J = (f, f_x, f_y)$ (primary for $\mathfrak{m} = (x,y)$)

<u>To guess the versal deformation space of $f(x,y) = 0$:</u>

<u>Seen</u> [Corollary 5.2] Let $R = k[x,y]$, $I = (f)$, $B = R/I$

$\Rightarrow$ Deformation of B over the dual numbers are in correspondence with $T^1(B/k, B)$.

<u>In this case we can compute it explicitely</u>!

$T^1(B/k, M) = \operatorname{coker}(\operatorname{Hom}(\Omega_{A/k}, M) \to \operatorname{Hom}(I/I^2, M))$

$I/I^2 \xrightarrow{\delta} \Omega_{A/k} \otimes_A B$ $\qquad$ $I/I^2 = (\bar{f})$

$\bar{b} \mapsto db \otimes 1$ $\qquad$ $\bar{f} \mapsto f_x dx + f_y dy$

$$\begin{array}{ccc} \operatorname{Hom}(\Omega_{A/k}, M) & \xrightarrow{\delta^*} & \operatorname{Hom}(I/I^2, M) \\ \varphi & \mapsto & \varphi \circ \delta \end{array}$$

$\varphi(dx) = m_1$
$\varphi(dy) = m_2$

$$\varphi(\delta(\bar{f})) = \varphi(f_x\,dx + f_y\,dy) = f_x m_1 + f_y m_2$$

Taking coker; $T^1(B/k, M) \simeq M/(f_x, f_y)M$

$$\Rightarrow T^1(B/k, B) = B/(f_x, f_y) = R/J.$$

So, take $g_1, \to g_r \in R$ whose images ~~generate R/J~~ form a basis of R/J as vect. space.

Take r new variables $t_1, \to t_r$ and define a deformation $\mathfrak{X}$ over $T = \operatorname{Spec} k[t_1, \to t_r]$ by

$$F(x,y,t) = f(x,y) + \sum_{i=1}^{r} t_i g_i(x,y) = 0$$

Thm: Given $f(x,y) = 0$ isolated plane sing, the deformation $\mathfrak{X}/T$ above is miniversal in the following sence:

(a) For any other def. $\mathfrak{X}'/S$, $S = \operatorname{Spec}\hat{R}$, $\hat{R}$ complete local ring, $\exists\ \varphi : S \to T$ st $\mathfrak{X}'$ and $\mathfrak{X} \times_T S$ become isomorphic after completing along the closed fiber over zero, and

(b) although φ is not unique, the induced map on Zariski tangent spaces of S and T is uniquely determined.

Example: Let $f(x,y) = y^2 - x^3$ (cusp)

$f_x = -3x^2$, $f_y = 2y$ (assume char $k \neq 2,3$)

$\Rightarrow k[x,y]/(f, f_x, f_y)$ is generated by 1 and x as vector space.

Let's consider

$$F(x,y;t,u) = y^2 - x^3 + t + ux = 0$$

(miniversal deformation)

For general t, u the system

$$\begin{cases} F = 0 \\ F_x = -3x^2 + u \\ F_y = 2y \end{cases}$$

has no solution (the curve is smooth)

But $F_y = 0 \Rightarrow y = 0$ $(*)_1$

$$\left.\begin{array}{l} (*)_1 + F = 0 \Rightarrow t = -ux + x^3 \\ F_x = 0 \Rightarrow u = 3x^2 \end{array}\right\} \begin{cases} t = -2x^3 \\ u = 3x^2 \end{cases}$$

$\Rightarrow$ They are singularities over $\Delta = (27t^2 - 4u^3 = 0)$.