Functors of Artin rings

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k: field of characteristic 0
$\mathcal{C}$: category of Artin local k-algebras
Goal: study deformation problems as functors $\mathcal{C} \to \mathrm{Set}$

Examples

- If X is a scheme over k
  $\mathrm{Def}_X : A \mapsto \{\text{deformations of } X \text{ over } A\} / \text{isomorphism}$
  see the chapter "miniversal and universal deformations of a scheme" for a detailed study

- If $R = k[[X_1, \dots, X_n]] / (F_1, \dots, F_n)$
  defines a functor $A \mapsto \mathrm{Hom}(R, A)$
  $= A \mapsto \{ (x_1, \dots, x_n) \in (\underset{\substack{\uparrow \\ \text{maximal ideal of } A}}{m})^n \mid \forall i = 1 \dots n \ \ F_i(x_1, \dots, x_n) = 0 \}$
  In fact here Hom denotes morphisms of local k-algebras, so that the images of $X_1, \dots, X_n$, which are in the maximal ideal of $k[[X_1, \dots, X_n]]$, must be in m. This is well defined since m is nilpotent and you can replace $X_1, \dots, X_n$ in power series by elements of a nilpotent ideal.

- If X is a scheme of finite type over k and x a k-rational point, it defines a functor
  $F_{X,x} : A \mapsto \{ f : \mathrm{Spec}(A) \to X \mid f_{|\mathrm{Spec}(k)} = x \}$.

If we take affine coordinates around x we have

$$\mathcal{O}_{X,x} \simeq \left(k[X_1,\dots,X_n]/(f_1,\dots,f_n)\right)_{(X_1,\dots,X_n)} \leftarrow \text{localization}$$

completion $$\widehat{\mathcal{O}}_{X,x} \simeq k[[X_1,\dots,X_n]]/(f_1,\dots,f_n)$$

$F_{X,x}$ is isomorphic to $A \mapsto \left\{ (x_1,\dots,x_n) \in (m)^n \;/\; \forall i=1\dots n \;\; f_i(x_1,\dots,x_n)=0 \right\}$.

In fact $f: \operatorname{Spec}(A) \to X$ is determined locally by $(x_1,\dots,x_n) \in A^n \;/\; f_i(x_1,\dots,x_n)=0$ and asking that $f|_{\operatorname{Spec}(k)} = x$ is asking $(x_1,\dots,x_n) \in (m)^n$ since x has coordinates 0.

~~So $F_{X,x}$~~

So $F_{X,x} \simeq A \mapsto \operatorname{Hom}(\widehat{\mathcal{O}}_{X,x}, A)$.

Pro-representable functors

We denote by $\widehat{\mathcal{C}}$: category of complete noetherian local k-algebras:

elements: $R = \varprojlim R/m^n$ and R/m^n is in $\mathcal{C}$

typically $R = k[[X_1,\dots,X_n]]/I$.

~~if $R = k[[X_1,\dots,X_n]]$ then $R/m^n = k[X_1,\dots,X_n]$~~

ex: if $R = k[[X]]$ then $R/m^n = k[X]/X^n$ is in $\mathcal{C}$

Def: A functor $F: \mathcal{C} \to \mathrm{Set}$ is called pro-representable if F is isomorphic to $A \mapsto \operatorname{Hom}(R, A)$ for some $R \in \widehat{\mathcal{C}}$.

Often a functor is not ~~pro~~-representable, but we are interested in morphisms of functor $\varphi : \mathrm{Hom}(R, -) \to F$.

Such a φ induces for all n $\quad \varphi_n : \mathrm{Hom}(R, R/m^n) \to F(R/m^n)$,

the image of $\pi_n : R \to R/m^n$ is denoted $\xi_n \in F(R/m^n)$.

The (ξ_n) form a <u>formal family</u>: the projection

$R/m^{n+1} \to R/m^n$ induces $F(R/m^{n+1}) \to F(R/m^n)$

that sends ξ_{n+1} ~~to~~ ξ_n, so $\xi_n \in \varprojlim F(R/m^n)$.

We denote by $\hat{F}(R)$ the category of formal families.

<u>Theorem</u> ("pro- Yoneda lemma")

There is a natural bijection

morphism $\mathrm{Hom}(R, -) \to F \quad \Leftrightarrow \quad \hat{F}(R)$

Proof: we have seen the map $\to$.

In the other direction, given (ξ_n) we define $\varphi : \mathrm{Hom}(R, -) \to F$:

if $A \in \mathcal{C}$ we will have $\varphi_A : \mathrm{Hom}(R, A) \to F(A)$.

if $f \in \mathrm{Hom}(R, A)$, f factors through $\pi_n : R \to R/m^n$ for $n \gg 0$.

$f = g\pi_n \quad g : R/m^n \to A$

and we let $\varphi_A(f) =$ image of $\xi_n \in F(R/m^n)$ by $F(g) : F(R/m^n) \to F(A)$

this doesn't depend on $n \gg 0$.

<u>Remark</u>: If $R \in \mathcal{C}$ (ie without all the R/m^n) this is exactly the content and the proof of the usual Yoneda lemma

$\mathrm{Hom}(R, -) \to F \quad \Leftrightarrow \quad F(R)$.

We say that the couple (R, ξ) pro-represents F.

Formal families
~~Couples~~ for a functor :

A ~~couple~~ formal family for a functor $F : \mathcal{C} \to Set$ will just be the data of $R \in \hat{\mathcal{C}}$ and $\xi \in \hat{F}(R)$, seen as a morphism $Hom(R, -) \to F$.

Def: a morphism of functor of Artin rings $\varphi : G \to F$ is strongly surjective if :

i. $\forall A \in \mathcal{C}$ $\varphi_A : G(A) \to F(A)$ is surjective and

ii. $\forall$ surjection $B \to A$ in $\mathcal{C}$ the induced morphism
$$G(B) \to G(A) \underset{F(A)}{\times} F(B)$$ is surjective.

Remarks. • The fiber product in ~~Sets~~, given $f : B \to A$ and $g : C \to A$, is $B \underset{A}{\times} C = \{ (b, c) \in B \times C \;/\; f(b) = g(c) \}$

The fibers product in the category $\mathcal{C}$ is induced by the one in Set.

Here the morphism $G(A) \to F(A)$ is induced by φ_A and $F(B) \to F(A)$ is induced by $F(B \to A)$.

• ~~If $F(k) = \{*\}$ singleton — which is not a big restriction, some authors put it in the definition of a functors of Artin rings — then ii $\Rightarrow$ i by taking $A = k$~~

• If $F(k)$ and $G(k)$ have only one element — which is not a big restriction, some authors put it in the definition of a functor of Artin rings — then ii $\Rightarrow$ i, simply with $A = k$, then $G(A) \underset{F(A)}{\times} F(B) = F(B)$

- A functor F is called <u>smooth</u> if $F(k)$ = singleton and $F \to *$ (constant functor) is smooth. This means $\forall$ surjection $B \to A$ in $\mathcal{C}$, $F(B) \to F(A)$ is surjective.

Recall the infinitesimal lifting property: this is exactly the case of the functor $F_{X,x}$ for a scheme X over k and x a smooth point!!

In case of $\varphi : \mathrm{Hom}(R, -) \to F$, property ii of strong surjectivity means:

$\overset{=}{G}$

$G(B) \twoheadrightarrow G(A) \times_{F(A)} F(B)$

given $g : B \to A$ a surjection in $\mathcal{C}$,
given a morphism $f : R \to A$ (ie $f \in G(A)$)
given an element $\theta \in F(B)$ that induces $\eta \in F(A)$ (via $F(g : B \to A)$)
such that f induces η via $\varphi_A : \mathrm{Hom}(R, A) \to F(A)$
then there exists some $h : R \to B$ (ie $h \in G(B)$) that $\underbrace{\text{induces } f}$ and θ

$\hookrightarrow$ this means that h is a lifting of f. $h = g \circ f$.

<u>Def</u> A ~~couple~~ formal family (R, ξ) for F is called
a. a <u>versal family</u> if $\mathrm{Hom}(R, -) \to F$ is strongly surjective
b. a <u>miniversal family</u> (or: a <u>hull</u>) if in addition $\mathrm{Hom}(R, D) \to F(D)$ is bijective $D = k[t]/t^2$.
c. a <u>universal family</u> if (R, ξ) pro-represents F.

Theorem Let (R, ξ) be a ~~[illegible]~~ formal family for F

a - If (R, ξ) is versal then $\forall (S, \eta)$ ~~[illegible]~~ other formal family, there is a ring morphism $f: R \to S$ that induces $\hat{F}(R) \to \hat{F}(S)$ which sends ξ to η.

b - If (R, ξ) is miniversal then the above f induces a unique map $R/_{m^2} \to S/_{m^2}$ (ie the derivative is unique).

c. If (R, ξ) is universal then f is unique.

Proof. a. Let $\varphi: \mathrm{Hom}(R, -) \to F$ determined by ξ.

For all n we have $\varphi_{S/m^n} : \mathrm{Hom}(R, S/_{m^n}) \to F(S/_{m^n})$

By surjectivity there is $\theta_n \in \mathrm{Hom}(R, S/_{m^n})$. $\eta_n \in$

such ~~that~~ that $\varphi_{S/m^n}(\theta_n) = \eta_n$.

By strong surjectivity we can lift the θ_n inductively to a compatible family: (see the previous notation)

given $S/_{m^{n+1}} \to S/_{m^n}$

given $\theta_n : R \to S/_{m^n}$ induced inductively ~~by~~ θ_n

given $\eta_{n+1} \in F(S/_{m^{n+1}})$ inducing $\eta_n \in F(S/_{m^n})$

there is $\theta_{n+1}: R \to S/_{m^{n+1}}$ that lifts θ_n.

This defines a map $R \to S = \varprojlim S/_{m^n}$

By construction it induces $\hat{F}(R) \to \hat{F}(S)$ and sends ξ to η.

b. It is enough to check that for all map $S/_{m^2} \to D$ the induced map by f $R/_{m^2} \to D$ is unique.

Indeed we think of $\bar{f}: R/_{m^2} \to S/_{m^2}$ as a derivative and a map $S/_{m^2} \to D$ as some coordinate of the derivative

(ex: $S = k[[X_1, X_n]]$
a map $S/_{m^2} \to D$ is induced by $X \mapsto X_i$)

But: $\mathrm{Hom}(R/_{m^2}, D) = \mathrm{Hom}(R, D)$
and $\mathrm{Hom}(R,D) \to F(D)$ is bijective.

c. It is ~~the~~ a corollary to the pro-Yoneda lemma:
η determines a unique map $\mathrm{Hom}(S, -) \to F$
but F is isomorphic to $\mathrm{Hom}(R,-)$
so η determines a unique map $\mathrm{Hom}(S,-) \to \mathrm{Hom}(R,-)$
which is induced by a unique $R \to S$.

<u>Schlessinger's criterion, necessary part</u>

We fix a functor $F: \mathcal{C} \to \mathrm{Set}$.
We denote by $t_F := F(D)$ the <u>tangent space</u> to F;
If X is a scheme over k with a k-point x then for $F_{X,x}$ the tangent space is exactly the usual Zariski tangent space.

We give a list of necessary conditions: and the full Schessinger's criterion at next lecture.

- If F has a versal family[a], then $F(k)$ has just one element.

Proof: $\mathrm{Hom}(R,k) \to F(k)$ is surjective and $\mathrm{Hom}(R,k)$ has just one element.

b) and: $\forall$ morphism $A' \to A$ and $A'' \to A$ in $\mathcal{C}$ the natural map
$$F(A' \times_A A'') \to F(A') \times_{F(A)} F(A'') \quad \text{is surjective.}$$
Proof by strong surjectivity (only part i is needed?)
elements of $F(A)$, $F(A')$, $F(A'')$ are induced by morphisms, $R \to A$, $R \to A'$, $R \to A''$ in a compatible way in order to define a map $R \to A' \times_A A''$, which then gives an element of $F(A' \times_A A'')$, all in a compatible way.

Furthermore if F has a universal family

c - $\forall A \in \mathcal{C}$, for the maps $A \to k$ and $D \to k$, the above
$$F(A \times_k D) \to F(A) \times_{F(k)} F(D) \quad \text{is bijective}$$

d. t_F has a natural structure of k-vector space, finite dim
Proof: $t_F = F(D) \simeq Hom(R, D)$, which is a k-vector space and we can pull-back this structure

e. $\forall p : A' \to A$ small extension and $\eta \in F(A)$, there is a transitive group action of the vector space t_F on $F(p)^{-1}(\eta)$ (if nonempty)

If F is pro-representable, then.

f. All the maps of b. are bijective and the action of e is simply transitive.

Proofs: see the book.