The Schlessinger's criterion and applications

- $k = \bar{k}$, $\mathcal{C}$ = category of local artinian k-alg. with residue field k.
  $\hat{\mathcal{C}}$ = category of complete local k-alg with residue field k.
- R local ring $\varphi : R \to \hat{R} = \varprojlim R/m^n$, $\ker \varphi = \bigcap m^n$
  so, if $R \in \mathcal{C} \Rightarrow R \simeq \hat{R} \Rightarrow \mathcal{C} \hookrightarrow \hat{\mathcal{C}}$.
- $R \in \hat{\mathcal{C}}$ define $h_R : \mathcal{C} \to \mathrm{Sets}$, $A \mapsto \mathrm{Hom}_k(R, A) = \mathrm{Hom}_{\mathrm{Spec}\,k}(\mathrm{Spec}\,A, \mathrm{Spec}\,R)$

<u>Recall</u>: $F : \mathcal{C} \to \mathrm{Sets}$ is pro-representable if it is isomorphic to h_R for some $R \in \hat{\mathcal{C}}$. $\boxed{\text{Clear}}$ F representable $\Rightarrow F$ pro-representable

<u>Ex</u>: X scheme$/k$, $x \in X(k)$

$$F(A) = \left\{ \begin{array}{ccc} \mathrm{Spec}(k) & \xrightarrow{x} & X \\ \downarrow & \nearrow_{\varphi} & \\ \mathrm{Spec}\,A & & \end{array} \right\} \quad \Rightarrow \quad F \simeq h_{\hat{\mathcal{O}}_{X,x}}.$$

<u>Pro-Yoneda</u>: There is a bijection

$$\mathrm{Hom}(h_R, F) \longleftrightarrow \Big\{ \text{elements of } \underbrace{\varprojlim F(R/m^n)}_{= \hat{F}(R)} \Big\}$$

Then, if $F : \mathcal{C} \to \mathrm{Sets}$ is prorepresentable
$\Rightarrow \exists\, \xi : h_R \xrightarrow{\sim} F$ for some $R \in \hat{\mathcal{C}}$
By Pro-Yoneda, we can think $\xi \in \hat{F}(R)$
We say that (R, ξ) pro-represents F.

Recall: Let $F : \mathcal{C} \to \text{Sets}$ be a functor. A pair (R, ξ), where $R \in \hat{\mathcal{C}}$ and $\xi \in \varprojlim F(R/m^n)$, is called

(a) Versal if $h_R \xrightarrow{\xi} F$ satisfies

(i) $\forall A \in \mathcal{C}$, $\text{Hom}_k(R, A) \twoheadrightarrow F(A)$

(ii) $\forall B \twoheadrightarrow A$ in $\mathcal{C}$ we have that: given a map $R \to A$, inducing $\eta \in F(A)$, and given $\theta \in F(B)$ mapping to η, one can lift the map $R \to A$ to a map $R \to B$ inducing θ.

} strongly surjective

(b) Miniversal if it is versal and $h_R(k[t]/t^2) \to F(k[t]/t^2)$ is bijective.

(c) Universal if it pro-represents the functor F.

Theorem (Schlessinger): The functor $F : \mathcal{C} \to \text{Sets}$ has a miniversal family if and only if:

(H_0) $F(k)$ has just one element.

(H_1) For every surjection $A'' \twoheadrightarrow A$ in $\mathcal{C}$ whose kernel I is 1-dim k-vector space (ie, a small extension), the map

$$F(A' \times_A A'') \twoheadrightarrow F(A') \times_{F(A)} F(A'')$$ is surjective.

(H_2) $F(A' \times_k k[t]/t^2) \to F(A') \times_{F(k)} F(k[t]/t^2)$ is bijective.

(H_3) $F(k[t]/t^2)$ is a finite-dim k-vector space.

Furthermore, F is pro-representable if and only if in addition:

(H_4) For every $\varphi : A'' \twoheadrightarrow A$ with $\dim_k \ker \varphi = 1$ and every $\eta \in F(A)$ for which $F(\varphi)^{-1}(\eta) \neq \emptyset$, the group action of t_F on $F(\varphi)^{-1}(\eta)$ is bijective.

(Here $t_F := F(k[t]/t^2)$)

Remark: $(H_4) \Rightarrow (H_2)$

<u>Pro-representability of some functors</u>:

Def: Let $F : \underline{Sch}_k \longrightarrow Sets$ be a contravariant functor. Given $X_0 \in F(\operatorname{Spec} k)$, we define its local functor $\widetilde{F} : \mathcal{C} \to Sets$ by

$$A \longmapsto \{ X \in F(\operatorname{Spec} A) \,/\, X \times_A k = X_0 \}$$

$\nwarrow$ abuse of notation

We will see (§ 23) that F representable $\Rightarrow \widetilde{F}$ pro-representable and that the converse is false.

a) <u>Local Hilb functor</u>: Let $X_0 \subseteq \mathbb{P}^n_k$ be a given closed subscheme and define $\widetilde{F} : \mathcal{C} \to Sets$ to be

$$A \longmapsto \{ \mathbb{P}^n_A \supseteq X \longrightarrow \operatorname{Spec} A \text{ flat} \,/\, X \times_{\operatorname{Spec} A} \operatorname{Spec} k \simeq X_0 \}$$

<u>Claim</u>: $\widetilde{F}$ is pro-representable: We apply Schlessinger's criterion

(H₀) $\widetilde{F}(k) = \{X_0\}$ one element ✓

(H₁) Let $A'' \twoheadrightarrow A$ small extension, $A' \to A$ any map.

We must show that

$$\widetilde{F}(A' \times_A A'') \to F(A') \underset{F(A)}{\times} F(A'')$$

is surjective.

So let $X' \subseteq \mathbb{P}^n_{A'}$ flat over $\operatorname{Spec} A'$ and $X'' \subseteq \mathbb{P}^n_{A''}$ flat over $\operatorname{Spec} A''$, both restricting to $X \subseteq \mathbb{P}^n_A$.

Consider the scheme $(X^*, \mathcal{O}_{X^*}) = (X, \mathcal{O}_{X'} \times_{\mathcal{O}_X} \mathcal{O}_{X''})$

and let $A^* = A' \times_A A''$

From

$$\begin{tikzcd} & \mathcal{O}_{\mathbb{P}^n_{A^*}} \arrow[dl] \arrow[dr] & \\ \mathcal{O}_{X'} \arrow[dr] & \circlearrowleft & \mathcal{O}_{X''} \arrow[dl] \\ & \mathcal{O}_X & \end{tikzcd}$$

we obtain $\mathcal{O}_{\mathbb{P}^n_{A^*}} \twoheadrightarrow \mathcal{O}_{X^*}$

$$\begin{tikzcd} & \mathcal{O}_{\mathbb{P}^n_{A^*}} \arrow[dl] \arrow[dr] & \\ \mathcal{O}_{\mathbb{P}^n_{A'}} & & \mathcal{O}_{\mathbb{P}^n_{A''}} \end{tikzcd}$$

$\Rightarrow X^*$ is a closed subscheme of $\mathbb{P}^n_{A^*}$, is flat over $\operatorname{Spec} A^*$ by base change, and restricts to $\mathcal{O}_{X'}$ and $\mathcal{O}_{X''}$ over $\operatorname{Spec} A'$ and $\operatorname{Spec} A''$ (technical result of the previous section)

Thus $X^* \in \widetilde{F}(A^*)$ shows that (H_1) is satisfied.

(H_3) $t_{\widetilde{F}} = \widetilde{F}(k[t]/t^2) \simeq H^0(X_0, N_{X_0/\mathbb{P}^n_k})$

which is finite dimensional since X_0 is projective.

(H_4) $(\Rightarrow (H_2))$: Let $\eta \in \widetilde{F}(A)$ be a given deformation $X \subseteq \mathbb{P}^n_A$ of X_0. Given $A' \overset{\varphi}{\twoheadrightarrow} A$ small, $\widetilde{F}(\varphi)^{-1}(\eta)$ is the set of subschemes $X' \subseteq \mathbb{P}^n_{A'}$, flat over $\operatorname{Spec} A'$, with $X' \times_{\operatorname{Spec} A'} \operatorname{Spec} A \simeq X$.

Seen: If such exists, they form a torsor under $H^0(X_0, N_{X_0/\mathbb{P}^n_k}) = t_{\widetilde{F}}$ ✓

$\Rightarrow \widetilde{F}$ is pro-representable ■

b) <u>Local Picard functor</u>: Let $X_0 \in \underline{Sch}_k$ and $\mathcal{L}_0 \in \operatorname{Pic}(X_0)$. We define $\widetilde{F}: \mathcal{C} \to \text{Sets}$ by

$$A \longmapsto \left\{ \begin{array}{l}\text{Isom. classes of}\\ \text{inv. sheaves}\end{array} \mathcal{L} \to X = X_0 \times_{\operatorname{Spec} k} \operatorname{Spec} A \;\middle/\; \mathcal{L} \otimes \mathcal{O}_{X_0} \simeq \mathcal{L}_0 \right\}$$

abuse of language.

If, X_0 is projective and $H^0(X_0, \mathcal{O}_{X_0}) = k$ then $\widetilde{F}$ is pro-representable:

(H_0) $\widetilde{F}(k) = \{\mathcal{L}_0\}$ ✓

(H_1) Let $\mathcal{L}' \to X'$ and $\mathcal{L}'' \to X''$ st $\mathcal{L}' \otimes \mathcal{O}_X \simeq \mathcal{L}'' \otimes \mathcal{O}_X \simeq \mathcal{L}$ on X. Choose ~~isomorphisms~~ map $\mathcal{L}' \to \mathcal{L}$, $\mathcal{L}'' \to \mathcal{L}$ inducing the isom.

Take $\mathcal{L}^* = \mathcal{L}' \times_{\mathcal{L}} \mathcal{L}'' \longrightarrow X^* = X_0 \times_{\operatorname{Spec} k} A^*$; $A^* = A' \times_A A''$

(Technical results of §16) $\Rightarrow$ It restricts to $\mathcal{L}'$ on X' and $\mathcal{L}''$ on $X'' \Rightarrow (H_1)$ ✓

(H_3) Seen: $t_{\tilde{F}} = \tilde{F}\left(k[t]/t^2\right) \simeq H^1(X_0, \mathcal{O}_{X_0})$

which is finite-dim if X_0 is projective ✓

(H_4) $(\Rightarrow (H_2))$

Seen (Thm 6.4) If $H^0(\mathcal{O}_{X_0}) = k$ then isom. classes of $\mathcal{L}'$ is a torsor under the action of $H^1(\mathcal{O}_{X_0})$. ✓

$\Rightarrow \tilde{F}$ is pro-representable ■