# Minimal and universal deformations of schemes

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Goal: apply Schlessinger's theory to the functor of deformations of a given scheme.

## Recall: The Schlessinger's criterion

Let $F : \mathcal{C} \to \text{Sets}$ a functor of Artin rings.
$F$ has a miniversal family $\iff$

$(H_0)$ $F(k)$ is a singleton

$(H_1)$ $\forall$ map $A' \to A$, $\forall$ small extension $A'' \to A$
$F(A' \times_A A'') \to F(A') \times_{F(A)} F(A'')$ is surjective

$(H_2)$ This map is a bijection if $A'' = D$, $A = k$

$(H_3)$ $t_F := F(D)$ is a finite dim $k$-vector space.

$F$ is prorepresentable $\iff$ in addition

$(H_4)$ $\forall$ small extension $p : A'' \to A$ and $\eta \in F(A)$ there is a simply transitive group action of $t_F$ on $F(p)^{-1}(\eta)$ (if nonempty).

## Application to existence of a miniversal family

We fix a scheme $X_0$ over $k$ and we consider the functor $F : \mathcal{C} \to \text{Sets}$

$$A \mapsto \left\{ \begin{array}{l} \text{schemes } X \text{ flat}/A \text{ with a given closed immersion} \\ i : X_0 \to X \text{ such that } i_0 \otimes_A k : X_0 \xrightarrow{\sim} X \times_A k \end{array} \right\}$$

mod isomorphism

Theorem 1. $F$ has a miniversal family $\iff$ one of the the following hypothesis holds

- $X_0$ is affine with isolated singularities
- $X_0$ is projective.

Proof

$(H_0)$ is clear and works because we quotient by isomorphisms

$$\left( \text{i.e. } \begin{tikzcd} & X' \\ X_0 \arrow[ur] \arrow[dr] & \\ & X'' \arrow[u, "\wr"] \end{tikzcd} \right)$$

In fact $F(k)$ consists of $X_0 \xrightarrow{id} X_0$.

An other deformation of $X_0$ ~~we~~ over $k$ would be an automorphism $\sigma : X_0 \to X_0$, and this is isomorphic to $id : X_0 \to X_0$ via $\sigma$

$$\begin{tikzcd} & X_0 \\ X_0 \arrow[ur, "\sigma"] \arrow[dr, "id"'] & \\ & X_0 \arrow[u, "\wr\ \sigma"'] \end{tikzcd}$$

$(H_1)$ given any $A' \to A$ and a small extension $A'' \to A$ and deformations $X' \in F(A')$, $X'' \in F(A'')$ which extend a deformation $X \in F(A)$ of $X_0$

$$\begin{tikzcd} X_0 \arrow[r, hook] \arrow[d] & X \arrow[r, hook] \arrow[dr, hook] \arrow[d] & X' \arrow[d] \\ & & X'' \arrow[d] \\ k \arrow[r] & A \arrow[r] \arrow[dr] & A' \\ & & A'' \end{tikzcd}$$

Define $X^*$: the support is $X$ ($= X_0$ - same support) and the sheaf is $\mathcal{O}_{X^*} = \mathcal{O}_{X'} \times_{\mathcal{O}_X} \mathcal{O}_{X''}$.

This gives a well-defined scheme over $A' \times_A A''$ which is a deformation of $X_0$, extending the deformations $X'$ and $X''$.

Remark: It seems that this construction is symmetric in $A'$ and $A''$, but the hypothesis need any map $A' \to A$ and a small extension $A'' \to A$. This is due to the technical lemmas that we skipped, and is needed to construct the scheme $X^*$.

($H_2$). Suppose $A = k$, $A'' = D$. ~~With~~ Then $X = X_0$.

~~Let $W$ be the scheme previously constructed (called X~~

Let $X^*$ be constructed as before, let $W$ be another scheme with the same properties.

By the universal property of fibered product, there is a map $X^* \to W$, both as deformations of $X$, so

$$X^* \times_k D \to W \times_k D \text{ is an isomorphism,}$$

and both are flat over $D$. By an already seen lemma, this implies that $X^* \to W$ is an isomorphism (as deformations ~~of~~ of $X$). So $X^*$ is unique.

($H_3$) Only here we need one of the two hypothesis.

. If $X_0$ is affine, then $X_0 = \operatorname{Spec}(B)$ $t_F = T^1_{B/k}$ and has support only at a finite number of singular points. This implies (??) finite length and finite dim.

In the general case there is an exact sequence

$$0 \to H^1(X_0, T_{X_0}) \to \mathrm{Def}(X_0/k, D) \to H^0(X_0, T^1_{X_0}) \to H^2(X, T_X).$$

(recall: we proved that $H^1(X_0, T_{X_0})$ parametrizes global deformations first-order of a smooth scheme. In fact we didn't need "smooth" but only "locally trivial". $T^1$ parametrizes deformations of a non-smooth, but affine, scheme. So $H^0(X_0, T^1_{X_0})$, as seen in Čech cohomology, describes obstruction to local deformations).

If $X_0$ is projective, then $H^1(X_0, T_{X_0})$ and $H^0(X_0, T^1_{X_0})$ are finite-dimensional, so also $\mathrm{Def}(X_0/k, D) = t_F$ □

<u>Application to pro-representability</u>

Theorem 2. (same notations) $F$ ~~is prorepre~~ is prorepresentable iff ($X_0$ satisfies the previous hypothesis) ~~and~~ $\forall A' \to A$ small extension, ~~and~~ a deformation $X$ over $A$ and an extension $X'$ over $A'$, the natural map $\mathrm{Aut}(X'/X_0) \to \mathrm{Aut}(X/X_0)$ is surjective.

Theorem 3. This happens if $X_0$ is a projective scheme and $H^0(X_0, T_{X_0}) = 0$.

~~Proof of theorem 3: by ~~artist~~ Artin induction~~

~~(to prove it~~

Proof of theorem 3. In fact we prove that forall deformation $X$ over $A$ $\mathrm{Aut}(X/X_0) = \{id\}$.
We prove it by Artin induction, i.e. for $A = k$ and we suppose that if it is true for $A$, then it is true for a small extension $A' \to A$. This implies ... true forall $A$

Case $A = k$: $\mathrm{Aut}(X_0/X_0) = \{id\}$ trivially.
Case of $A' \to A$, $X$ deformation$/A$, assume $\mathrm{Aut}(X/X_0) = \{id\}$
Let $X'$ be an extension of the deformation $X$ to $A'$.
Let $\sigma' \in \mathrm{Aut}(X'/X_0)$. Then $\sigma'$ induces $\sigma \in \mathrm{Aut}(X/X_0)$
~~=~~ so $\sigma = id$, and $\sigma'$ induces an element in $\mathrm{Aut}(X'/X)$. But we saw that the set of such automorphism is classified by $H^0(X_0, T_{X_0})$ ($\otimes J = \mathrm{Ker}(A' \to A)$) which is $0$. So $\sigma'$ must be id. $\square$

Proof of theorem 2, ~~Si~~ See the book.
~~Since F has a miniversal family there is already a transitive gray action of $t_F$ on the fibers of $F(A') \to F(A)$; we only need to show that it is simply transitive~~

~~Suppose $\mathrm{Aut}(X'/X_0) \to \mathrm{Aut}(X/X_0)$ surjective $\forall X'/X$~~
~~let $X'_1, X'_2 \in F(A')$ both in the same fiber, i.e. are extensions of a deformation $X/A$ of $X_0$.~~
~~Suppose $X'_1 = X'_2$ as elements of $F(A')$, i.e. there is a map $u: X'_1 \xrightarrow{\sim} X'_2$ over $X_0$~~