Comparison of Embedded and Abstract Deformations

Aim: Compare deformations of a scheme X_0 as a closed subscheme of $\mathbb{P}^n$ to its deformations as an abstract scheme.

Formally, if F_1 is the functor of Artin rings of embedded deformations and F_2 is the functor of abstract deformations, then we have a "forgetful morphism" $F_1 \longrightarrow F_2$ forgetting the embedding.

Prop: Let $f : F_1 \to F_2$ be a morphism of functors of Artin rings. Assume that F_1 and F_2 both have versal families corresponding to complete local rings R_1, R_2.

$\Rightarrow \exists\, \bar{f} : \operatorname{Spec} R_1 \longrightarrow \operatorname{Spec} R_2$ corresponding to $\varphi : R_2 \to R_1$ such that for each Artin ring A

$$\begin{tikzcd} \operatorname{Hom}(R_1, A) \arrow[r, "\varphi^*"] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & \operatorname{Hom}(R_2, A) \arrow[d] \\ F_1(A) \arrow[r, "f"] & F_2(A) \end{tikzcd} \quad \Big\} \text{ versal families.}$$

is commutative. Furthermore, if R_1 and R_2 are <u>miniversal</u> then the map induced by $\bar{f}$, $t_{R_1} \to t_{R_2}$ is just
$t_{F_1} \to t_{F_2}$
$F_1(D) \to F_2(D)$.

Proof: $R_1 \to R_1/\mathfrak{m}^n$ induce elements $\xi_n \in F_1(R_1/\mathfrak{m}^n)$.
By f we get $f(\xi_n) \in F_2(R_1/\mathfrak{m}^n)$.
By the versal property of R_2 we get $R_2 \longrightarrow R_1/\mathfrak{m}^n$
and hence a homomorphism $R_2 \longrightarrow \varprojlim R_1/\mathfrak{m}^n = R_1$ ∎

Let $X_0 \subseteq \mathbb{P}^n_k$ projective closed subscheme, and let F_1 and F_2 be the functor of embedded deformations and abstract deformations, resp. $f: F_1 \to F_2$ forgetful morphism.

<u>Seen</u>: F_1 is pro-representable $\rightsquigarrow R_1 \in \hat{\mathcal{C}}$

F_2 has a miniversal family $\rightsquigarrow R_2 \in \hat{\mathcal{C}}$.

[Prop] $\Rightarrow \exists \bar{f}: \operatorname{Spec} R_1 \to \operatorname{Spec} R_2$ associated morphism.

<u>Prop</u>: Suppose that $X_0 \subseteq \mathbb{P}^n$ is nonsingular. Then

$$0 \to T_{X_0} \to T_{\mathbb{P}^n}|_{X_0} \to N_{X_0/\mathbb{P}^n} \to 0$$

gives an exact sequence

$$0 \to H^0(T_{X_0}) \to H^0(T_{\mathbb{P}^n}|_{X_0}) \to H^0(N_{X_0/\mathbb{P}^n}) \xrightarrow{\delta^0} H^1(T_{X_0})$$

$$\to H^1(T_{\mathbb{P}^n}|_{X_0}) \to H^1(N_{X_0/\mathbb{P}^n}) \xrightarrow{\delta^1} H^2(T_{X_0}) \to H^2(T_{\mathbb{P}^n}|_{X_0}) \to \dots$$

in which $\delta^0: H^0(N_{X_0/\mathbb{P}^n}) \to H^1(T_{X_0})$ is the induced maps on tangent spaces $t_{F_1} \to t_{F_2}$, and $\delta^1: H^1(N_{X_0/\mathbb{P}^n}) \to H^2(T_{X_0})$ maps the obstruction space of F_1 to the obstruction space of F_2.

<u>Remark</u>: By the Proposition, we can interpret

- $H^1(T_{\mathbb{P}^n}|_{X_0})$: obstructions to lifting an abstract deformation of X_0 to an embedded deformation of X_0
- $\operatorname{im}(H^0(T_{\mathbb{P}^n}|_{X_0}) \to H^0(N_{X_0/\mathbb{P}^n}))$: deformations of X_0 induced by automorphisms of $\mathbb{P}^n$.

<u>The case of smooth surfaces in $\mathbb{P}^3$</u>: Let $d = \deg(X) \geq 2$.

By restricting $0 \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus(n+1)} \to T_{\mathbb{P}^n} \to 0$ (Euler)
(with $n=3$) to X we obtain:

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(1)^4 \to T_{\mathbb{P}^3}|_X \to 0$$

and

$$\begin{aligned} 0 &\to H^0(\mathcal{O}_X) \to \bigoplus_{i=1}^{4} H^0(\mathcal{O}_X(1)) \to H^0(T_{\mathbb{P}^3}|_X) \\ &\to H^1(\mathcal{O}_X) \to \bigoplus_{i=1}^{4} H^1(\mathcal{O}_X(1)) \to H^1(T_{\mathbb{P}^3}|_X) \\ &\to H^2(\mathcal{O}_X) \to \bigoplus_{i=1}^{4} H^2(\mathcal{O}_X(1)) \to H^2(T_{\mathbb{P}^3}|_X) \to 0 \end{aligned}$$

Suppose $k = \mathbb{C}$, then by Hodge theory and Lefschetz hyperplane theorem: $h^{0,0}(X) = h^{0,0}(\mathbb{P}^3) = 1$

$$q(X) = h^{1,0}(X) = h^{0,1}(X) = h^{1,0}(\mathbb{P}^3) = 0$$
$$p_g(X) = h^{0,2}(X) = h^{2,0}(X)$$

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ p_g & & h^{1,1} & & p_g \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

(Hodge diamond)

<u>Known</u>: $0 \to \mathcal{O}_{\mathbb{P}^3}(-d) \to \mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_X \to 0$

$\Rightarrow$ Hilbert polynomial of X: $P_X(n) = \chi(X, \mathcal{O}_X(n)) = \binom{n+3}{3} - \binom{n+3-d}{3}$

$$= \frac{1}{2} d n^2 + \frac{1}{2}(4d - d^2)\, n + \frac{1}{6}(d^3 - 6d^2 + 1$$

For $n = 0$:

$$\chi(X, \mathcal{O}_X) = \underbrace{h^0(\mathcal{O}_X)}_{1} - \overset{0}{\cancel{h^1(\mathcal{O}_X)}} + h^2(\mathcal{O}_X) = \frac{1}{6}(d^3 - 6d^2 + 11d)$$

$$\Rightarrow h^{2,0}(X) = h^{0,2}(X) = \frac{(d-1)(d-2)(d-3)}{6}$$

<u>Remark</u>: Noether's formula: $\chi(X, \mathcal{O}_X) = \frac{1}{12}\left(K_X^2 + \chi_{top}(X)\right)$

where $\chi_{top}(X) = \sum_{i=0}^{4} (-1)^i \underbrace{h^i(X, \mathbb{C})}_{b_i} = 2 + b_2$, $K_X = (d-4) H_X$, $H_X^2 = d$

$$\Rightarrow K_X^2 = d(d-4)^2 \Rightarrow \chi_{top}(X) = 12\,\chi(X, \mathcal{O}_X) - d(d-4)^2$$

$$\Rightarrow h^{1,1}(X) = \frac{d(2d^2 - 6d + 7)}{3}.$$

$\Rightarrow h^0(T_{\mathbb{P}^3}|_X) = 4h^0(X, \mathcal{O}_X(1)) - 1 = 16 - 1 = 15$

$h^1(T_{\mathbb{P}^3}|_X) = \begin{cases} 1, & d=4 \\ 0, & d \neq 4 \end{cases}$

$h^2(T_{\mathbb{P}^3}|_X)$ can be computed from

$$4\chi(X, \mathcal{O}_X(1)) = \chi(X, \mathcal{O}_X) + \chi(X, T_{\mathbb{P}^3}|_X)$$

and $\begin{cases} 0, & d \leq 5 \\ \neq 0, & d \geq 6 \end{cases}$.

Remark: $\mathcal{I}_X/\mathcal{I}_X^2 \simeq \mathcal{O}_X(-d) \Rightarrow N_X \simeq \mathcal{O}_X(d)$.

$\Rightarrow H^0(T_{\mathbb{P}^3}|_X) \longrightarrow H^0(N_X)$ is surjective for $d=2$ and injective for $d \geq 3$.

Since $h^1(N_X) = h^1(\mathcal{O}_X(d)) = 0 \quad \forall d \geq 2$, we can construct the following table

d	$h^0(T_X)$	$h^0(T_{\mathbb{P}^3}\|_X)$	$h^0(N_X)$	$h^1(T_X)$	$h^1(T_{\mathbb{P}^3}\|_X)$
2	6	15	9	0	0
3	0	15	19	4	0
4	0	15	34	20	1
≥ 5	0	15	large	large	0

If $d=2$, $h^1(T_X)=0$ so there are no abstract deformations (i.e. it is rigid). $X \simeq \mathbb{P}^1 \times \mathbb{P}^1 \Rightarrow \mathrm{Aut}(X) \simeq PO(4) \simeq PGL(2) \times PGL(2) \rtimes S_2$ is of dimension 6. There is a 9-dim. family of quadric surfaces in $\mathbb{P}^3$, any two related by an automorphism of $\mathbb{P}^3$.

If $d \geq 3$, $h^0(T_X) = 0$, so there are no infinitesimal automorphisms $\Rightarrow F_2 : \mathcal{C} \to \mathrm{Sets}$ (abs. defo.) is pro-representable.

Except for $d=4$, every deformation of X is realized as a deformation inside $\mathbb{P}^3$:

- F_1 has no obstructions $(h^1(N_X)=0)$
- $t_{F_1} \to t_{F_2}$ surjective

$\overset{\text{Ex. 15.8}}{\Rightarrow}$ $F_1 \to F_2$ is strongly surjective.

Remark: $h^2(T_X) = 0$ for $d \leq 5$ (abstract deformations are unobstructed)

But $h^2(T_X) \neq 0$ for $d \geq 6$. Even so, abstract deformations are unobstructed (Ex. 20.1)

The case $d=4$: In this case X is a K3 surface

Here: • $h^1(\mathcal{N}_X) = 0$ ($\mathcal{F}_1$ is unobstructed)

• $\mathcal{F}_1$ is prorepresentable $\longrightarrow R_1 \in \hat{\mathcal{C}}$.

• $h^0(\mathcal{N}_X) = 34 \Rightarrow \dim_{Krull} R_1 = 34$

• $\mathcal{F}_2$ is also unobstructed and pro-representable $\rightsquigarrow R_2 \in \hat{\mathcal{C}}$ with $\dim_{Krull} R_2 = h^1(T_X) = 20$

~~and~~ $\operatorname{Spec} R_1 \rightarrow \operatorname{Spec} R_2$ induce $t_{R_1} \rightarrow t_{R_2}$ which is not surjective; its image has dimension $34 - 15 = 19$!

So there are abstract deformations of X_0 that cannot be realized as embedded deformations in $\mathbb{P}^3$

Remark

1) Over $\mathbb{C}$, there are complex K3 surfaces that are not algebraic!

2) In a similar way, if A is an abelian variety $/\mathbb{C}$ of dimension $g > 1$.

$\Rightarrow h^1(A, T_A) = g^2$, but $\dim A_g = \dfrac{g(g+1)}{2}$

"Half of its deformations are not algebraic"!