# Introduction to moduli questions

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## Goal

$\mathcal{M}$ = sets of objects to be classified, modulo isomorphism

ex: closed subschemes of $\mathbb{P}^n_k$; curves of genus $g$;
curves of genus 0 ($\mathcal{M}$ = singleton)
curves of genus 0 with a marking of three distinct points ($\mathcal{M}$ = singleton $\mathbb{P}^1 + 0, 1, \infty$)

We want to construct a scheme $M/k$ such that
closed points of $M \Leftrightarrow$ elements of $\mathcal{M}$

ex: curves of genus 0 $\to M$ = point
closed subschemes of $\mathbb{P}^n_k \to M$ = Hilbert scheme.

Suppose we have a family $\begin{matrix} X \\ \downarrow \\ S \end{matrix}$ such that $\forall s \in S$
the fiber $X_s$ is an element of $\mathcal{M}$

ex: $\begin{matrix} \mathbb{P}^n_k \times S \\ \downarrow \text{ flat} \\ S \end{matrix}$ for $\mathcal{M} = \{\text{closed subschemes of } \mathbb{P}^n_k\}$

a $\mathbb{P}^1$-bundle $\begin{matrix} X \\ \downarrow \\ S \end{matrix}$ is a family for $\mathcal{M} = \{\mathbb{P}^1\}$.

Then this determines a map $f : S \to M$

$s \in S \mapsto$ point of $M$ corresponding to $X_s$ as element of $\mathcal{M}$.

This map is functorial with respect to families:

If $S' \to S$ and $\begin{array}{c} X \\ \downarrow \\ S \end{array}$ is a family, then there is a pull-back family $\begin{array}{c} X' \\ \downarrow \\ S' \end{array}$ and the map $S' \to M$ is the composition $S' \to S \to M$.

So: we have a functor $\mathcal{F}_e : (\mathrm{Schemes}/k)^{op} \to \mathrm{Sets}$

$S \mapsto \{\text{families over } S\}$ (modulo isomorphism that we have to define).

$(S' \to S) \mapsto$ pull back of a family over $S$ to a family over $S'$

and a moduli space $M$ must define a natural transformation

$\varphi : \mathcal{F}_e \to \mathrm{Hom}(-, M)$

$S \mapsto$ map $S \to M$.

<u>Moduli spaces</u>

We suppose we have defined $\mathcal{M}$ and a notion of family and of isomorphisms. So we work ~~not~~ abstractly with a functor $\mathcal{F}_e$. $(\mathrm{Schemes}/k)^{op} \to \mathrm{Sets}$, associated to $\mathcal{M}$.

Remark: in a family $\begin{array}{c} X \\ \downarrow \\ S \end{array}$ the fiber over $s \in S$ is the pull-back family by $s : \mathrm{Spec}(k) \to S$ thus it is $\mathcal{F}_e(\mathrm{Spec}(k) \to S)\left(\begin{array}{c} X \\ \downarrow \\ S \end{array}\right)$ if $\begin{array}{c} X \\ \downarrow \\ S \end{array} \in \mathcal{F}_e(S)$

Def. A *coarse* moduli space for $\mathcal{F}$ is a scheme $M/k$ with a morphism of functors $\varphi: \mathcal{F} \to \mathrm{Hom}(-, M)$ such that

i. $\varphi_k: \mathcal{F}(k) \to \mathrm{Hom}(k, M)$ is bijective

ii if $\psi: \mathcal{F} \to \mathrm{Hom}(-, N)$ is another morphism, then $\exists\, e: M \to N$ such that $\psi = e_* \circ \varphi$

Remark: i $\mathcal{F}(k) = \{\text{families over } k\} = \text{elements of } \mathcal{M}$ so it says $\mathcal{M} \leftrightarrow$ points of $M$.

ii says $M$ is minimal, the closest to represent the functor $\mathcal{F}$.

By functoriality for $\mathcal{F}(S)$ and $s: \mathrm{Spec}(k) \to S$ the "fiber over $s$" for a family $X \in \mathcal{F}(S)$, which is $\mathcal{F}(s: \mathrm{Spec}(k) \to S)(X)$, is the point of $M$ corresponding to the pull-back family of $X$ by $s$, ie the fiber $X_s$.

Def A *tautological family* for a coarse moduli $M$ is a family $X/M$ such that $\forall m \in M$ the fiber $X_m$ is exactly the element of $\mathcal{M}$ corresponding to $m$, ie: it is a family $\begin{array}{c} X \\ \downarrow \\ M \end{array} \in \mathcal{F}(M)$ which is sent by

$\varphi_M: \mathcal{F}(M) \to \mathrm{Hom}(M, M)$ to $\mathrm{id}_M$.

It may not exist.

Def a <u>fine</u> moduli ~~scheme~~ space for $\mathcal{F}$ is a scheme $M/k$ with $\varphi : \mathcal{F} \to \mathrm{Hom}(-, M)$ which is an isomorphism (ie $\mathcal{F}$ is representable).

In this case $\mathcal{F}$ has a tautological family (we call it universal family $X_u$) which is
$\varphi^{-1}(\mathrm{id}_M : M \to M) \in \mathcal{F}(M)$. It has an additional property:
for all family $\begin{matrix} X \\ \downarrow \\ S \end{matrix}$ (ie $X \in \mathcal{F}(S)$), it is obtained in a unique way as a pullback by the map $S \to M$ of the universal family $X_u$:

Proof: ~~$X \in \mathcal{F}(S)$ $X \in \mathcal{F}(S)$ induces by $\varphi_S$ an all element element $f: S \to M$~~

It's just a version of Yoneda. Given $X \in \mathcal{F}(S)$, defines a map $f: S \to M$, draw the diagram

$$\begin{tikzcd} \varphi_M^{-1}: & \mathrm{Hom}(M, M) \arrow[r] \arrow[d, "f^*"'] & \mathcal{F}(M) \arrow[d, "\mathcal{F}(f)"] \\ \varphi_S^{-1}: & \mathrm{Hom}(S, M) \arrow[r] & \mathcal{F}(S) \end{tikzcd}$$

Now $\mathrm{id}_M$ is sent (top and right) to $\mathcal{F}(f)(X_u)$
and (left and bottom) to $\varphi_S^{-1}(f^* \mathrm{id}_M) = \varphi_S^{-1}(f) = X$
so $X = \mathcal{F}(f)(X_u)$, which represents the pullback.

~~The existence of $X_{u/M}$~~

Conversely, given $M$, ~~then~~ if it has such a universal family $X_u$ then $M$ represents $\mathcal{F}$.

Property: If $\varphi: \mathcal{F} \to \mathrm{Hom}(-, M)$ is a fine moduli space, then it is a coarse one.

Proof ~~$\mathcal{F}(k) \to$~~ $\mathcal{F}(k) \to \mathrm{Hom}(k, M)$ is a bijection because $\varphi$ is an isomorphism of functors;
If $\psi: \mathcal{F} \to \mathrm{Hom}(-, N)$, then $\psi \cdot \varphi^{-1}: \mathrm{Hom}(-, M) \to \mathrm{Hom}(-, N)$, by Yoneda it is induced by some $e: M \to N$
and $\psi = e_* \varphi$.

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Def: A family $\begin{matrix} X \\ \downarrow \\ S \end{matrix}$ is <u>trivial</u> if it is obtained by base extension from a family over a point.
A family $\begin{matrix} X \\ \downarrow \\ S \end{matrix}$ is <u>fiberwise trivial</u> if all fibers $X_s$ ~~are~~ give the same element of $\mathcal{M}$.

Proposition: If $\mathcal{M}$ has a fine moduli space:
~~triv~~ fiberwise trivial $\Rightarrow$ trivial

Proof $\begin{matrix} X \\ \downarrow \\ S \end{matrix}$ defines a map $S \to M$ which is ~~constants~~ constant and we have seen that $\begin{matrix} X \\ \downarrow \\ S \end{matrix}$ is obtained by pulling back the universal family by this map, ie pulling back a family over a point.

So the condition of having a fine moduli space is strong.

Example (from differential geometry)

- $\mathcal{M} =$ vector spaces of rank 1 over $\mathbb{R}$
  has just one point.
  family over $\mathcal{M} =$ line bundle
  There are non trivial line bundles! But: fiberwise trivial
- $\mathcal{M} =$ vector spaces of rank 1 over $\mathbb{R}$ + base point $\neq 0$
  also has just one point
  family = line bundle + section everywhere $\neq 0$
  Fiberwise trivial and globally trivial!!

This shows that sometimes, in order to have a fine moduli space we must add marked points, "rigidify" the structure.

<u>Local study of the moduli space</u>

Theorem. Let $M$ fine moduli scheme. Let $X_0 \in \mathcal{M}$ corresponding to $x_0 \in m$.
Then: Zariski tangent space at $x_0 \leftrightarrow$ families $X$ over $D$ whose fiber over $k$ is isomorphic to $X_0$.

Proof. already seen for the Hilbert scheme!
Both $\leftrightarrow$ maps $\in$ ~~Hom(k~~ $\mathrm{Hom}(D, M)$ that send $k$ to $x_0$.

Theorem. Given $\mathcal{M}$, $\mathcal{F}$. ~~let $\mathcal{F}_0$ be~~

Fix $X_0 \in \mathcal{M}$. Let $\mathcal{F}_0$ be the functor that to each local $k$-algebra $A$ assigns the set of families over $A$ whose fiber over $k$ is isomorphic to $X_0$.

If $\mathcal{M}$ has a fine moduli scheme, then $\mathcal{F}_0$ is pro-representable.

Proof. Suppose ~~M~~ $M$ is a fine moduli scheme.

$X_0 \in \mathcal{M} \iff x_0 \in M$.

~~$\mathcal{F}_0($ elements of $\mathcal{F}_0(A) \iff$ morph~~

elements of $\mathcal{F}_0(A)$ which reduces to $X_0$

$\iff$ morphisms $A \to M$ that send $k$ to $x_0$.

and those are pro-represented by $\widehat{\mathcal{O}}_{M,x_0}$, the completed local ring

In ~~the~~ some sense, deformation theory is the study of local structure of moduli spaces!

Warning: In families, $\mathcal{F}_0$ is ~~the~~ not the usual functor of deformation, ~~because the isomorphism~~

$$\begin{array}{ccc} X_0 & \to & X \\ \downarrow & & \downarrow \\ k & \to & A \end{array}$$

because the isomorphism with $X_0$ is not induced by a fixed map $X_0 \to X$.

But we can cogue it (see Hartshorne, theorem 18.4)

Theorem: ~~If a scheme as a fi~~

If an element of $\mathcal{M}$ has a finite group of automorphism, then there is no fine moduli space.

Theorem. $M = \mathrm{Spec}(k)$ is a coarse moduli space for curves of genus 0 and has a tautological family

Theorem. Consider $\mathcal{M}$ = curves of genus 0 with a chosen point; a family is $\begin{matrix} X \\ \downarrow \\ S \end{matrix}$ with a section $+$ flat, whose fibers are isomorphic to $\mathbb{P}^1_k$.

Then $M = \mathrm{Spec}(k)$ is a coarse moduli space, there is a ~~tautol~~ tautological family, and every family is locally trivial.

Theorem. Consider $\mathcal{M}$ = curves of genus 0 with three distinct marked points; family = $\begin{matrix} X \\ \downarrow \\ S \end{matrix}$ flat with three everywhere distinct sections.

Then $\mathcal{M}$ has a fine moduli space.