Some Representable Functors

We have already stated:

<u>Thm</u>: The Hilbert functor, which to every scheme S/k associates the set of subschemes $Y \subseteq \mathbb{P}^N_S$, flat over S, whose fibers all have a given Hilbert polynomial P, is <u>representable</u> by a scheme $\mathrm{Hilb}^P(X)$, projective over k.

In the course of the proof, one has to stablish certain properties of the functor:

<u>Def</u>: A contravariant functor $\mathcal{F} : \mathfrak{S}ch_k \to \mathrm{Sets}$ is

- <u>bounded</u>: if $\exists$ S scheme of finite type$/k$ and $X \in \mathcal{F}(S)$ a family st every $X_0 \in \mathcal{F}(k)$ is isomorphic to the fiber X_s for some $s \in S$ closed point.

  (<u>Here</u>: X_s is the image of X in $\mathcal{F}(k)$ induced by
  $$\begin{array}{ccc} \mathrm{Spec}\, k & \longrightarrow & S \\ \{*\} & \longmapsto & \overset{\in}{s} \end{array}$$
)

- <u>separated</u>: $\forall$ C/k non singular curve and $p_0 \in C$, if $X, X' \in \mathcal{F}(C)$ st $X_p \simeq X'_p$ $\forall p \in C \setminus \{p_0\}$ <u>then</u>, $X_{p_0} \simeq X'_{p_0}$.

- <u>complete</u>: $\forall$ C/k non singular curve and $p_0 \in C$, given $X \in \mathcal{F}(C \setminus \{p_0\})$, $\exists$ $X' \in \mathcal{F}(C)$ st $X_p \simeq X'_p$ $\forall p \neq p_0$.

<u>Prop</u>: If $\mathcal{F}$ is represented by M of finite type$/k$ then $\mathcal{F}$ is bounded. In that case, M separated (resp. proper)$/k$ $\Leftrightarrow$ $\mathcal{F}$ separated (resp. separated and complete)

To show that the Hilbert functor is bounded, one use Castelnuovo-Mumford regularity:

Def: A coherent sheaf $\mathcal{F}$ on a proj. scheme X is <u>m-regular</u> if $H^i(\mathcal{F}(m-i)) = 0 \quad \forall i > 0$.

(Here: $\mathcal{F}(k) = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(k)$.)

Prop (Mumford): If $\mathcal{F}$ is m-regular, then $\mathcal{F}$ is m'-reg $\forall m' \geq m$. Furthermore $\mathcal{F}(m)$ is gen. by global sections.

Prop: A family $\mathcal{M}$ of coherents sheaves on a proj. scheme X/k, all having the same Hilbert pol., is bounded (ie, the functor of flat families of sheaves in $\mathcal{M}$ is bounded) iff $\exists$ uniform m_0 st all members of $\mathcal{M}$ are m_0-regular.

Proof: (outline)

($\Rightarrow$) $\mathcal{M}$ bounded $\Rightarrow \exists\, S/k$ of finite type and $\mathcal{F} \in \underline{\mathrm{Coh}}(X \times S)$, flat$/S$, containing among its fibers at closed points of S all elements of $\mathcal{M}$.

$\forall\, i, n \quad h^i(\mathcal{F}_s(n))$ is u.s. continuous for $s \in S$. (flatness)

By looking at the generic points of all irred. components of S $\rightsquigarrow n_0$ uniform st $h^i(\mathcal{F}_s(n)) = 0 \quad \forall n \geq n_0$ at those generic points.

By semicont, $h^i = 0$ on a dense open subset $\mathcal{U} \subseteq S$.

Let $S_1 = S - \mathcal{U} \rightsquigarrow$ find n_1 st $h^i(\mathcal{F}_s(n)) \ \forall n \geq n_1$ at the generic points of $S_1 \rightsquigarrow \mathcal{U}_1 \subseteq S_1$ dense $\rightsquigarrow S_2 = S_1 - \mathcal{U}_1$

$\Rightarrow \mathcal{F}_s$ is $(\max\{n_i\} + \dim X)$-regular.

($\Leftarrow$) Suppose $\exists m_0$ st all elements in $\mathcal{M}$ are m_0-regular

$\Rightarrow \mathcal{F}(m_0)$ is globally generated and $h^0(\mathcal{F}(m_0)) = P(m_0)$ ← Hilbert pol.

(since $h^i = 0$ for $i > 0$)

$\Rightarrow \exists\ \mathcal{O}_X^N \to \mathcal{F}(m_0)$ for each $\mathcal{F} \in \mathcal{M}$, $N = h^0(\mathcal{F}(m_0))$

Let $\mathcal{G}$ be the kernel.

Long exact seq. in cohomology $\Rightarrow \exists m_0'$ uniform st all $\mathcal{G}$ are m_0'-regular. Hence $\mathcal{G}(m_0')$ is globally gen.

Let $M = h^0(\mathcal{G}(m_0'))$

$\Rightarrow \mathcal{F}$ is determinated by the M-dim subspace $H^0(\mathcal{G}(m_0'))$ of $H^0(\mathcal{O}_X^N(m_0'))$

$\leadsto$ Parametrized by a finite-dim grassmanian Gr

Over a suitable subspace of ~~the~~ Gr we obtain a family containing all of our initial sheaves $\mathcal{F}$. ∎

By using this proposition and induction on the dimension (by taking generic hyperplane sections) it can be shown:

__Prop__: The set of subschemes Y of $\mathbb{P}^n_k$ with Hilbert polynomial P form a bounded family.

__Remark__: $\mathrm{Hilb}^P(\mathbb{P}^n_k)$ is separated and complete follows, because if C is a curve and $U \subseteq C$ open, and if $Y \subseteq \mathbb{P}^n_U$ is a closed subscheme flat$/U$, then $\exists!$ $\bar{Y} \subseteq \mathbb{P}^n_C$ flat$/C$ restricting to Y: the scheme-theoretic closure of Y in $\mathbb{P}^n_C$.

Remark: We can associate to $Y \subseteq X = \mathbb{P}^n_k$ closed subscheme
$\rightsquigarrow \mathcal{O}_Y$. So $\mathrm{Hilb}^P(X)$ parametrize all quotients
$\mathcal{O}_X \to \mathcal{O}_Y \to 0$ with Hilbert pol. P.

Generalization: Fix $\mathcal{E} \in \underline{\mathrm{Coh}}(X)$, then $\mathrm{Quot}_{\mathcal{E}}(X)$ parametrize all quotients $\mathcal{E} \to \mathcal{F} \to 0$, and $\mathcal{F}$ runs through all quotients with given Hilbert pol.
Grothendieck showed that $\mathrm{Quot}_{\mathcal{E}}(X)$ is representable by a proj. scheme.

Other variant: the Hilbert-flag scheme parametrizing nested sets of closed subschemes. E.g., for a flag of length 2, fix $X = \mathbb{P}^n_k$ and P, Q be Hilbert pols. We consider the functor $\mathcal{F}$ that for each $S \in \underline{\mathrm{Sch}}_k$ assigns a pair of closed subschemes $Y \subseteq Z \subseteq X \times S$, both flat$/S$, and where the fibers of Y (resp. Z) have Hilbert pol. P (resp. Q)
$\Rightarrow \mathcal{F}$ is represented by a proj. scheme.

We can also deform a given morphism of schemes$/k$ $f: X \to Y$ (keeping X, Y fixed). A deformation of f over an Artin ring A is a morphism $f': X \times A \to Y \times A$ st $f' \otimes k = f$.

Lemma: To give a defo. of $f: X \to Y$ is equivalent to give a defo. of the graph Γ_f as a closed subscheme of $X \times Y$.

<u>Proof</u>: To any defo f' of f we associate its graph
$\Gamma_{f'} \subseteq X \times Y \times A$ $\rightsquigarrow$ a defo. of Γ_f.
Conversely, given a defo. Z of Γ_f over A, we need to verify that it is a graph of some morphism.
The projection $p_1 : Z \to X \times A$ gives an isomorphism when tensored with k.
Z flat$/A$ $\Rightarrow$ p_1 is an isomorphism
$\Rightarrow Z$ is the graph of $f' = p_2 \circ p_1^{-1}$. ∎

<u>Prop</u>: Assume that Y is non-singular. Then the tangent space to the deformation functor of $f : X \to Y$ is $H^0(X, f^*T_Y)$, and the obstruction to deforming f lie in $H^1(X, f^*T_Y)$.
If X and Y are also projective, the deformation functor is also pro-representable.

<u>Proof</u>: We must consider deformations of Γ_f as a closed subscheme of $X \times Y$.
Note that $\Gamma_f = (f \times id)^{-1}(\Delta_Y)$, where $\Delta_Y \subseteq Y \times Y$ is the diagonal

$$\overset{\shortparallel}{\{(x,y) \in X \times Y \,/\, y = f(x)\}}$$

Y non singular $\Rightarrow \Delta_Y$ l.c.i. in $Y \times Y$ (locally given by $\dim Y$ equations)
and $\mathcal{I}_\Delta / \mathcal{I}_\Delta^2 = \Omega^1_{Y/k}$ (via the isom $\Delta \simeq Y$)

$\Rightarrow \Gamma_f$ is a l.c.i. with normal bundle f^*T_Y. ∎

$$0 \to N_f^\vee \to f^*\Omega^1_Y \to \Omega^1_X \to \Omega^1_{X/Y} \to 0$$

It follows that if X and Y are projective, Y nonsingular, and $f: X \to Y$ a morphism. Then, locally around $[f]$, the scheme $\mathrm{Mor}(X,Y)$ can be defined by $h^1(X, f^*T_Y)$ equations in a nonsingular variety of dimension $h^0(X, f^*T_Y)$.

In particular, any irred. component of $\mathrm{Mor}(X,Y)$ through $[f]$ has dimension at least

$$h^0(X, f^*T_Y) - h^1(X, f^*T_Y)$$

<u>Important case</u>: $f: \mathbb{P}^1 \longrightarrow X$, X smooth projective.

Here: $\dim_{[f]} \mathrm{Mor}(\mathbb{P}^1, X) \geq \chi(\mathbb{P}^1, f^*T_X)$

<u>Riemann-Roch</u>: If C is a nonsingular curve of genus g and $\mathcal{E}$ is a vector bundle, then:

$$\chi(C, \mathcal{E}) = \mathrm{rk}(\mathcal{E}) \underbrace{\chi(\mathcal{O}_C)}_{(1-g)} + \underbrace{\deg(\mathcal{E})}_{\substack{\deg(\det \mathcal{E}) \\ \shortparallel \\ C \cdot \det \mathcal{E}}}$$

<u>Here</u>: $C \simeq \mathbb{P}^1$, $\mathcal{E} = f^*T_X$

$$\Rightarrow \dim_{[f]} \mathrm{Mor}(\mathbb{P}^1, X) \geq n - \mathbb{P}^1 \cdot f^*K_X = n - K_X \cdot f_*\mathbb{P}^1.$$

<u>Spirit</u>: If K_X is "negative", X has a lot of rational curves!